修　士　論　文　の　和　文　要　旨

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論　文　題　目	Uncertainty relation from parameter estimation of the position of an electron in a uniform magnetic field		

要　　旨

１９６０年代頃から量子推定理論に基づいて分散を求めた結果と量子力学における不確定性関係との比較をした研究がなされてきた。その結果、量子推定理論に基づく結果がHeisenberg-Robertson の不確定性関係と同様な関係を与えることが知られていた。一方、渡辺らは d 次元のフルパラメータモデルについて量子推定理論が Heisenberg-Robertson の不確定性関係とは異なる結果を与えることを示した。

本研究では，一様磁場中にある一つの電子という物理的に意味のある具体的なモデルを設定した。量子推定理論により一様磁場中にある電子の位置座標(x, y)における x と y の分散の下界を求め，不確定性関係について論じた。渡辺らの先行研究は有限次元におけるものであり本質的に無限次元となるモデルについて調べているところが本研究と先行研究の違いである。

一様磁場中にある電子の位置に関する不確定性関係を量子推定理論に基づいて求めるにあたり本研究では，推定に使うユニタリー変換として正準運動量と力学的運動量を生成子とした場合の二つを用いて量子推定を行った。前者をモデル 1，後者をモデル 2 と呼ぶ。モデル 1，モデル 2 ともに量子推定理論を用いると Heisenberg-Robertson の不確定性関係（Δx）（Δy）≧0 よりタイトな下界が得られることが示せた。

モデル１では熱状態（混合状態）を基準状態とした時，Gaussian Shift モデルと同等になることがわかった。従って RLD クラメール・ラオ不等式が与える下界が SLD のものよりタイトな下界となり，また RLD の与える下界は達成可能であることもわかった。基底状態（純粋状態）を基準状態としてとった場合は，正準運動量の x，y 成分が非可換なことを反映し，SLD フィッシャー情報行列に量子的な効果が見られた。

一方のモデル２で熱状態（混合状態）を基準状態とした時Δx, Δy のとりうる領域は RLD, SLD の与える下界，Z 行列の与える上界の組み合わせで表される複雑な構造になることがわかった。さらにその下界の形は電子の角運動量に依存して変わるという興味深い結果が得られた。基底状態（純粋状態）を基準状態とした場合は，力学的運動量の x, y 成分が可換なことを反映し，SLD フィッシャー情報行列に準古典的な効果が見られた。

平成３０年度（２０１８年度）　修士論文

Uncertainty relation from parameter estimation of the position of an electron in a uniform magnetic field

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Chapter 1

Introduction

1.1 Background

The uncertainty relation based on the quantum estimation theory was investigated by many authors [1, 2, 4, 5]. It is known that the one-parameter unitary model with a pure reference state, the Heisenberg-Robertson uncertainty relation and the uncertainty relation by the parameter estimation have the same form, see for example [2, 4]. In literature, many authors discussed similarity between two-different types of uncertainty relations. In Ref. [5], they showed that the uncertainty relation for a generic full parameter qudit model can be different when derived from the quantum parameter estimation theory.

1.2 About the present work

In the present work, we set up a specific physical model, a model of one electron in a uniform magnetic field and investigate the uncertainty relation of the position of the electron. We used the parameter estimation of two-parameter unitary model and compare the result to the Heisenberg-Robertson uncertainty relation. One of the reasons to analyze a specific model is that it should be worthwhile to have knowledge about the quantum mechanical limit before designing an experiment to examine any quantum mechanical effect. This work can be a good example of the use of quantum estimation theory for designing an experiment.

In this model, the Heisenberg-Robertson uncertainty relation of the position of an electron

$(x,\ y)$ only yields to the following trivial inequality.

$$(\Delta x)(\Delta y) \geq \frac{1}{2}|\langle[x, y]\rangle| = 0. \tag{1.1}$$

This is because two position operators x and y commute, i.e., $[x, y] = 0$. More detail about the Heisenberg-Robertson uncertainty relation is given in Appendix A.2. In the (1.1), Δx denotes the standard deviation about x with respect to a state ρ, which is defined by

$$(\Delta x)^2 = \mathrm{tr}\,[\rho(x - \langle x\rangle)^2] = \langle x^2\rangle - \langle x\rangle^2, \tag{1.2}$$

where $\langle x\rangle_\rho = \mathrm{tr}\,[\rho\, x]$ and $\langle x^2\rangle_\rho = \mathrm{tr}\,[\rho\, x^2]$. Therefore, $\langle x\rangle_\rho$ and $\langle x^2\rangle_\rho$ are the expectation value of x and x^2, respectively. Δy is defined in the same way.

We estimate the position of the electron in a uniform field based on the quantum estimation theory to see if the quantum estimation theory gives the result different from the Heisenberg-Robertson uncertainty relation. We used two kinds of the unitary transformations to estimate the change in the electron position by the transformations. In one model (Model 1), the unitary transformation of which generator is the mechanical momentum is used and in the other (Model 2), the transformation with the generator of the canonical (ordinary) momentum is used. In Model 1, we need only one set of the creation-annihilation operators to describe the model. In Model 2, however, we need two sets of the creation-annihilation operators. With the thermal state as the reference state in both Model 1 and 2, we obtained the tighter bounds than (1.1).

In Model 1, when the reference state is the thermal state (mixed state), it turned out that this model is equivalent to the Gaussian shift model. Therefore, the RLD Cramer-Rao bound gives a tighter bound than the SLD Cramer-Rao bound does. It turned out that the RLD Cramer-Rao bound is an achievable bound. When the reference state is a pure state made of the ground state, Fisher information matrices indicate the transformation be quantum mechanical. This is because x and y component of the mechanical momentum do not commute.

In Model 2, when the reference state is the thermal state with the assumption that we can give, or control the average angular momentum, $\langle l_z\rangle$. The shape of the bound depends on $\langle l_z\rangle$ and it shows a discontinuous change at $\langle l_z\rangle = \frac{1}{2}$. At $\langle l_z\rangle = 0$, the SLD Cramer-Rao bound is an achievable bound. This result, most likely, results from the infinitely degenerate angular momentum. This is a good example for showing that the degeneracy in the system makes the bound more complex. It is quite interesting that a physical quantity such as angular momentum changes the shape of the bound.

When the reference state is a pure state made of the ground state, unlike Model 1, Fisher information matrices indicate the transformation be quasi-classical. This is because x and y component of the canonical momentum do commute.

1.3 About this thesis

First, in Chapter 1, the quantum estimation theory is reviewed. In Chapter 2, the quantum estimation method used in the present work is reviewed, following [4]. The present work is explained in Chapter 3. The detailed calculations are given in the Appendices.

Chapter 2

Quantum estimation theory

A quantum state (ρ) is positive ($\rho \geq 0$) and its trace is ($\mathrm{tr}\,\rho = 1$), therefore, the set of states is defined by

$$\mathscr{S}(\mathscr{H}) := \{\rho \in \mathscr{L}(\mathscr{H}) \,|\, \text{(i)}\, \rho \geq 0, \text{(ii)}\, \mathrm{tr}\,\rho = 1\}, \tag{2.1}$$

where $\mathscr{L}(\mathscr{H})$ is the set of all linear operators on $\mathscr{H}$. We need to define Qunatum Fisher Information when there is a family of states $M = \{\rho_\theta \,|\, \theta = (\theta^1, \cdots \theta^n) \in \Theta\} \subset \mathscr{S}(\mathscr{H})$. The question is, how do we define an inner product $\langle \cdot, \cdot \rangle_\rho$ and the score function $D_{\theta,i}$ as in the classical estimation theory? Unfortunately, there is no uniquely well-defined likelihood function in quantum system. This is because, for example, in a d-dimensional quantum system, operators A, B being $A, B \in \boldsymbol{C}^{d\times d}$ do not commute. We cannot define the multiplication uniquely. Because of this situation, Quantum score functions were defined as follows, depending on the order of their acting on the density matrix ρ_θ.

Definition $L^L_{\theta,i}$, $L^R_{\theta,i}$, $L^S_{\theta,i}$ [1, 2, 3]
$L^L_{\theta,i}$, $L^R_{\theta,i}$ and $L^S_{\theta,i}$ are defined by the solution to the following equations.

$$\frac{\partial \rho_\theta}{\partial \theta^i} = L^L_{\theta,i}\,\rho_\theta\,, \tag{2.2}$$

$$\frac{\partial \rho_\theta}{\partial \theta^i} = \rho_\theta\, L^R_{\theta,i}\,, \tag{2.3}$$

$$\frac{\partial \rho_\theta}{\partial \theta^i} = \frac{1}{2}\{\rho_\theta, L^S_{\theta,i}\} = \frac{1}{2}(\rho_\theta L^S_{\theta,i} + L^S_{\theta,i}\rho_\theta) \tag{2.4}$$

$L^L_{\theta,i}$, $L^R_{\theta,i}$ and $L^S_{\theta,i}$ are called left logarithmic derivative (LLD), right logarithmic derivative (RLD), and symmetric logarithmic derivative (SLD), respectively. In general, $L^\dagger_{\theta,i} = L_{\theta,i}$ but $(L^R_{\theta,i})^\dagger = L^L_{\theta,i} \neq L^R_{\theta,i}$ where $(L^R_{\theta,i})^\dagger$ represents the Hermit conjugate of $L^R_{\theta,i}$.

Definition ($\langle\cdot,\cdot\rangle_\rho$: inner product)

$$\langle A,B\rangle_\rho^R = \mathrm{tr}\,\rho BA^\dagger$$
$$\langle A,B\rangle_\rho^L = \mathrm{tr}\,\rho A^\dagger B$$
$$\langle A,B\rangle_\rho^S = \mathrm{tr}\,\frac{1}{2}\rho\,\{A^\dagger,B\} = \frac{1}{2}(\mathrm{tr}\,\rho A^\dagger B + \mathrm{tr}\,\rho BA^\dagger)$$

Definiton (Quantum Fisher information)

RLD quantum Fisher information $G^R(\theta) := g^R_{ij}(\theta)$ is

$$g^R_{ij}(\theta) := \langle L^R_{\theta,i}, L^R_{\theta,j}\rangle^R_{\rho_\theta} = \mathrm{tr}\,[\rho_\theta L^R_{\theta,j}(L^R_{\theta,i})^\dagger] = \mathrm{tr}\,[(\partial_i\rho_\theta)(L^R_{\theta,i})^\dagger] = \langle L^R_{\theta,i}, \partial_j\rho_\theta\rangle_{HS} = \langle \partial_i\rho_\theta, L^R_{\theta,j}\rangle_{HS}$$

where $\partial_i\rho_\theta = \dfrac{\partial\rho_\theta}{\partial\theta^i}$ and $\langle\cdot,\cdot\rangle_{HS}$ denotes the Hilbert-Schmidt inner product, $\langle A,B\rangle_{HS} = \mathrm{tr}\,[A^\dagger B]$.

$$\because\;\; \mathrm{tr}\,[\rho_\theta L^R_{\theta,j}(L^R_{\theta,i})^\dagger] = \mathrm{tr}\,[(L^R_{\theta,i})^\dagger\rho_\theta L^R_{\theta,j}] = \mathrm{tr}\,[(\rho_\theta L^R_{\theta,i})^\dagger L^R_{\theta,j}] = \mathrm{tr}\,[(\partial_i\rho_\theta)^\dagger L^R_{\theta,j}] = \langle\partial_i\rho_\theta, L^R_{\theta,j}\rangle_{HS} \quad (2.5)$$

Similarly, we define two more quantum Fisher information by

$$g^L_{ij}(\theta) := \langle L^L_{\theta,i}, L^L_{\theta,j}\rangle^L_\rho = \mathrm{tr}\,[\rho_\theta(L^L_{\theta,i})^\dagger L^L_{\theta,j}] = \mathrm{tr}\,[(L^L_{\theta,i})^\dagger L^L_{\theta,j}\rho_\theta] = \langle L^L_{\theta,i}, \partial_j\rho_\theta\rangle_{HS} = \langle\partial_i\rho_\theta, L^L_{\theta,j}\rangle_{HS}$$
$$g^S_{ij}(\theta) := \langle L_{\theta,i}, L_{\theta,j}\rangle_\rho = \mathrm{tr}\,[\frac{1}{2}\rho_\theta(L^\dagger_{\theta,i}L_{\theta,j} + L_{\theta,j}L^\dagger_{\theta,i})] = \mathrm{tr}\,[\frac{1}{2}(\rho_\theta L^\dagger_{\theta,i}L_{\theta,j} + L^\dagger_{\theta,i}\rho_\theta L_{\theta,j})]$$
$$= \mathrm{tr}\,[\frac{1}{2}\{\rho_\theta, L_{\theta,i}\}L_{\theta,j}] = \mathrm{tr}\,[(\partial_i\rho_\theta)L_{\theta,j}] = \langle\partial_i\rho_\theta, L_{\theta,j}\rangle_{HS}$$
$$G^S(\theta) := [g^S_{ij}(\theta)],$$
$$G^L(\theta) := [g^L_{ij}(\theta)]$$

$G^S(\theta)$ is called SLD quantum Fisher information and $G^L(\theta)$ is called LLD quantum Fisher information.

It is worth noting the followings.

1. $G^R(\theta),\ G^L(\theta),\ G^S(\theta) \geq 0$, i.e., they are positive matrices.
   $G^R(\theta),\ G^L(\theta)$ are Hermite and complex matrices.
   $G^S(\theta)$ Hermite and real matrix.
2. Since $(L^R_{\theta,i})^\dagger = L^L_{\theta,i}$, $L^L_{\theta,i}$ can be calculated from $L^R_{\theta,i}$.

$g^R_{ij}(\theta) = \mathrm{tr}\,[\rho_\theta L^R_{\theta,j}(L^R_{\theta,i})^\dagger] = \mathrm{tr}\,[\rho_\theta(L^L_{\theta,j})^\dagger L^L_{\theta,i}] = g^L_{ji}(\theta)$

2.1 Estimation of the position

2.1.1 Position and momentum operators

In quantum mechanics, the position and momentum are the self-adjoint operators Q an P which satisfy the commutation relation,

$$[Q, P] = QP - PQ = \mathrm{i}\hbar \tag{2.6}$$

where $\hbar$ is the Planck constant divided by 2π. Hereafter, we use the natural unit system which gives $\hbar = 1$ and $c = 1$ where c is the speed of light. In the Schrodinger representation, $Q = x$ and $P = -\mathrm{i}\dfrac{d}{dx}$

2.1.2 Unitary operator

We consider a unitary operator U_θ on $\mathscr{H} = L^2(\mathbb{R}^3)$. Let $\psi(\boldsymbol{r})$ be a complex function of the position $\boldsymbol{r} = (x, y, z)$. $L^2(\mathbb{R}^3)$ is the set of the functions $\psi(\vec{r})$ that satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(\boldsymbol{r})|^2 dxdydz < \infty \tag{2.7}$$

In Dirac's notation, $\psi(x) = \langle x\,|\,\psi\rangle$. Then, its state ρ is expressed as $\rho = |\psi\rangle\langle\psi|$. When $U_\theta = \mathrm{e}^{-i\theta p_x}$,

$$(U_\theta\psi)(x) = \mathrm{e}^{-i\theta p_x}\psi(x) = \mathrm{e}^{-i\theta(-\mathrm{i}\frac{d}{dx})}\psi(x) = \mathrm{e}^{-\theta\frac{d}{dx}}\psi(x) = \sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}\frac{d^n}{dx^n}\psi(x) = \psi(x-\theta), \tag{2.8}$$

Therefore, a new state ρ_θ can be generated from $\rho_0 = |\psi\rangle\langle\psi|$, $\rho_\theta = U_\theta\rho_0 U_\theta^\dagger$.

Let Q and P be the position and momentum operators. In Schrodinger representation, $Q = x$ and $P = -\mathrm{i}\frac{d}{dx}$ in one demensional case. The expectation value of Q with respect to the state ρ_θ is

$$\langle Q\rangle_\rho = \mathrm{tr}\,[\rho_\theta Q] = \mathrm{tr}\,[U_\theta\rho_0 U_\theta^\dagger Q] = \mathrm{tr}\,[\rho_0 U_\theta^\dagger Q U_\theta] = \langle Q\rangle_0 + \theta.$$

where $\langle Q\rangle_{\rho_\theta} = \mathrm{tr}\,[\rho_\theta Q]$ and $\langle Q\rangle_0 = \mathrm{tr}\,[\rho_0 Q]$.

$U_\theta^\dagger Q U_\theta = Q + \theta$ is derived from (2.8). θ is to be estimated under the assumption that ρ_0 is known. In the situation above, θ is estimated from the measured value of $Q - \langle Q\rangle_0$, that is, $x - \langle Q\rangle_0$. When the new operator, $T = Q - \langle Q\rangle_0$ is assumed to be unbaised, i.e., $\langle T\rangle_\rho = \theta$, its variance satisfies $\langle(T-\theta)^2\rangle_\rho = (\Delta T)^2 = (\Delta Q)^2$. Then, from the uncertainty relation,

$$\langle(T-\theta)^2\rangle_\rho \geq \frac{1}{4(\Delta P)^2}. \tag{2.9}$$

Since the unbiasedness gives $\frac{d}{d\theta}\langle T\rangle_\rho = 1$, we have $|\langle[T, P]\rangle_\rho| = 1$. Therefore, from the Heisenberg-Robertson uncertainty relation, $\Delta T\Delta P \geq \frac{1}{2}|\langle[T, P]\rangle_{\rho_\theta}| = \frac{1}{2}$. By using (2.9), we have

$$\Delta T\Delta P = \Delta Q\Delta P = \frac{1}{2} \tag{2.10}$$

2.1.3 Mixed state (Gaussian state)

We first define the creation-annihilation operators $A^\dagger$ and A by

$$A = \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP) \tag{2.11}$$

$$A^\dagger = \frac{1}{\sqrt{2c}}(Q - \mathrm{i}cP) \tag{2.12}$$

where Q and P are the position and the momentum operators for one dimensional motion, respectively and c is any real number. Since $[P,\ Q] = \mathrm{i} = \sqrt{-1}$ holds, $A^\dagger$ and A satisfies the commutation relation,

$$[A,\ A^\dagger] = AA^\dagger - A^\dagger A = 1 \tag{2.13}$$

Hamiltonian of a harmonic oscillator H is known to be expressed as

$$H = aP^2 + bQ^2 \tag{2.14}$$

with the constants a and b.

The coherent state $|z\rangle$ is defined as an eigenvector of A, i.e., $A\,|z\rangle = z\,|z\rangle$.

We define a Gaussian state by

$$S_{\kappa,z} = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,|z'\rangle\langle z'|\,d^2z', \tag{2.15}$$

where κ is a positive real number and $|z\rangle$ is the coherent state. Let the initial state, or the reference state be $\rho_0 = S_{\kappa,z}$. By the definition of the coherent state, $A\,|z\rangle = z\,|z\rangle$ and $\langle z|\,A^\dagger = \langle z|\,z^*$, the expectation and variance of Q, P are obtained as follows. (The calculation is given in Appendix B.)

$$(\Delta Q)^2 = \mathrm{tr}\,[\rho_0(Q - \langle Q\rangle_{\rho_0})^2] = \frac{c}{2}(1 + 4\kappa^2). \tag{2.16}$$

$$(\Delta P)^2 = \mathrm{tr}\,[\rho_0(P - \langle P\rangle_{\rho_0})^2] = \frac{1}{2c}(1 + 4\kappa^2). \tag{2.17}$$

When $\rho_0 = S_{\kappa,z}$, $\rho_\theta = U_\theta\rho_0U_\theta^\dagger$ is expressed as

$$\rho_\theta = S_{\kappa,\,z+\theta/\sqrt{2c}}. \tag{2.18}$$

(2.16) and (2.17) give $\Delta Q \Delta P = \frac{1}{2}(1+4\kappa^2)$. When $\kappa > 0$, $\Delta Q \Delta P$ cannot achieve the lower bound of $\Delta Q \Delta P \geq \frac{1}{2}$.

2.2 Parameter estimation

2.2.1 One-parameter estimation

A quantum statistical model is a family of quantum states $M = \{\rho_\theta \,|\, \theta \in \Theta\}$. Suppose we have an operator L_θ that satisfies the following equation,

$$\frac{d\rho_\theta}{d\theta} = \frac{1}{2}(\rho_\theta L_\theta + L_\theta^\dagger \rho_\theta). \tag{2.19}$$

Then,

$$\frac{d}{d\theta}\langle T \rangle_\rho = \mathrm{Re}(L_\theta, T - t)_\theta$$

where $(X, Y)_\theta = \langle Y, X^\dagger \rangle_\rho = \mathrm{tr}\,[\rho_\theta Y X^\dagger]$.

With using Schwartz's inequality,

$$\langle L_\theta,\, L_\theta^\dagger \rangle_\rho \langle T - \theta \rangle_\rho^2 \geq \mathrm{Re}\,(L_\theta, T - \theta)_\theta \tag{2.20}$$

From $\frac{d}{d\theta}\langle T \rangle_\rho = 1$ (T unbiased),

$$\mathrm{Re}(L_\theta, T - \theta)_\theta = 1. \tag{2.21}$$

We define $G(\theta)$ by

$$G(\theta) = \langle L_\theta,\, L_\theta^\dagger \rangle_\rho = \mathrm{tr}\,[\rho L_\theta L_\theta^\dagger] \tag{2.22}$$

Then, from (2.22) and (2.21), (2.20) is

$$\therefore \langle (T - \theta)^2 \rangle_\rho \geq \frac{1}{G(\theta)}. \tag{2.23}$$

(2.23) is called the quantum Cramer-Rao inequality, since it has a very similar form of the (classical) Cramer-Rao inequality.

If ρ_θ can be expressed as $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta X}\rho_0 \mathrm{e}^{-\mathrm{i}\theta X}$ with a Hermite operator X, then with $L_\theta = -2\mathrm{i}(X - \langle X \rangle_\rho)$, equation (2.19) holds. If $L_\theta = -2\mathrm{i}(X - \langle X \rangle_\rho)$, we have

$$\begin{aligned} G(\theta) &= \mathrm{tr}\,[\rho_\theta L_\theta L_\theta^\dagger] \\ &= 4\mathrm{tr}[\rho_\theta (X - \langle X \rangle_\rho)^2] \\ &= 4(\Delta X)^2 \end{aligned}$$

Therefore,

$$\langle (T-\theta)^2\rangle_\rho \geq \frac{1}{4(\Delta X)^2}. \tag{2.24}$$

When $X = P$, this is equivalent to (2.9). The current result, the result by the one-parameter quantum estimation gives the same result that the Heisenberg-Robertson uncertainty relation gives.

If L_θ ($L_\theta = L_\theta^\dagger$), as defined earlier, L_θ is SLD at a point θ. SLD, L^S_θ of a family of quantum states $M = \{\rho_\theta\}$ in $(C.7)$ is given as

$$L^S_\theta = \frac{2}{c(1+\kappa^2)}(Q - \langle Q\rangle_0). \tag{2.25}$$

Then,

$$G^S(\theta) = \frac{1}{g_\theta} = \frac{c(1+4\kappa^2)}{2}. \tag{2.26}$$

$$\langle (T-\theta)^2\rangle_\rho = (\Delta Q)^2 = \frac{c}{2}(1+4\kappa^2) = 1/G^S(\theta). \tag{2.27}$$

This is identical with (2.16). $\langle (T-\theta)^2\rangle_\rho = 1/G^S(\theta)$ holds.

Summary

The result by the one-parameter quantum estimation gives the same result that the Heisenberg-Robertson uncertainty relation gives.

2.2.2 Two-parameter estimation

Consider an unbiased estimator $T = (T_1, \cdots, T_n)$. The (j,k) component of the mean square error (MSE) matrix V_θ is defined by

$$[V_\theta]_{j,k} = \langle (T_j - \langle T_j\rangle_{\rho_\theta})(T_k - \langle T_k\rangle_{\rho_\theta})\rangle_{\rho_\theta} \tag{2.28}$$

The expectation value of estimator is

$$(\langle T_1\rangle_{\rho_\theta}, \cdots, \langle T_n\rangle_{\rho_\theta}). \tag{2.29}$$

The SLD Cramer-Rao inequality $V_\theta \geq (G^S(\theta))^{-1}$ holds and the RLD Cramer-Rao inequality $V_\theta \geq (G^R(\theta))^{-1}$ holds.

Consider a quantum statistical model $M_2 = \{\rho_\theta \,|\, \theta = (\theta^1, \theta^2) \in \mathbb{R}^2\}$ $\rho_\theta = U_{\theta^1} V_{\theta^2} \rho_0 U_{\theta^1}^\dagger V_{\theta^2}^\dagger$ where $U_{\theta^1} = \mathrm{e}^{-\mathrm{i}P\theta^1}$ and $V_{\theta^2} = \mathrm{e}^{\mathrm{i}Q\theta^2}$. When $\rho_0 = S_{\kappa,z}$, we obtain $\rho_\theta = S_{\kappa,z+(\theta^1+\mathrm{i}\theta^2)/\sqrt{2c}}$.

When we estimate θ^1 only, $T_1 = Q - \langle Q\rangle_{(0,\theta_2)}$ is the optimal estimator. When we estimate θ^2 only, $T_2 = P - \langle P\rangle_{(\theta^1,0)}$ is the optimal estimator. For this model, with using the mean square error (MSE) matrix V_θ, the RLD Cramer-Rao inequality which gives the tight bound is expressed

$$V_\theta \geq (G^R(\theta))^{-1} = \frac{1}{2}\begin{pmatrix} (1+4\kappa^2)c & -\mathrm{i} \\ \mathrm{i} & \dfrac{1+4\kappa^2}{c} \end{pmatrix}.$$

The calculation of $(G^R_\theta)^{-1}$ is given in Appendix C.3. We evaluate the inequality to derive the bound, or the uncertainty relation. Let $c = 1$ for simplicity. Then, by the same way used in Appendix D.6.4, we have

$$(V_{\theta,11})(V_{\theta,22}) \geq \frac{1}{4}(1+4\kappa^2+1)^2 = (1+2\kappa^2)^2$$

By definition,

$$V_{\theta,11} = (\Delta T_1)^2 = (\Delta Q)^2, \tag{2.30}$$

and

$$V_{\theta,22} = (\Delta T_2)^2 = (\Delta P)^2. \tag{2.31}$$

Therefore,

$$(\Delta P)^2(\Delta Q)^2 \geq (1+2\kappa^2)^2$$

$$(\Delta P)(\Delta Q) \geq 1+2\kappa^2 > 1 \tag{2.32}$$

In the meantime, from the Heisenberg-Robertson uncertainty relation, we obtain

$$(\Delta P)(\Delta Q) \geq \frac{1}{2}|\langle[P,\,Q]\rangle| = \frac{1}{2} \tag{2.33}$$

The uncertainty relation by the two-parameter estimation is larger than that by the Heisenberg-Robertson uncertainty relation. This difference results from measuring the two parameters at the same time.

Chapter 3

Uncertainty relation from parameter estimation of the position of an electron in a uniform magnetic field

3.1 Hamiltonian ($\hbar = 1,\ c = 1$)

We consider an electron in a uniform magnetic field. Hamiltonian of this model H is

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2. \tag{3.1}$$

where m and $-e$ are the mass and the charge of electron ($e > 0$), respectively. $\vec{A}$ is the vector potential.

We will investigate the uncertainty relation of an electron motion in a uniform magnetic field $\vec{B} = (0, 0, B)$, $B > 0$. We choose $\vec{A} = (-\frac{By}{2}, \frac{Bx}{2}, 0)$. With a new vector variable, $\vec{\pi} = \vec{p} + e\vec{A}$, $H = \frac{1}{2m}(\pi_x^2 + \pi_y^2 + p_z^2)$. Since z componet solution is a plane wave solution, we will consider the motion in $x - y$ plane only. Then, our Hamiltonian becomes [6]

$$H = \frac{1}{2m}(\pi_x^2 + \pi_y^2). \tag{3.2}$$

By using two independent sets of creation-annihilation operators, a, $a^\dagger$ and b, $b^\dagger$ [7], it is known that Hamiltonian H, z component of the angular momentum l_z, π_x, π_y, p_x, p_y, x, and y can be expressed as

$$H = \omega(a^\dagger a + \frac{1}{2}), \tag{3.3}$$

$$l_z = a^\dagger a - b^\dagger b, \tag{3.4}$$

$$\pi_x = \frac{\mathrm{i}}{\lambda}(a^\dagger - a), \tag{3.5}$$

$$\pi_y = \frac{1}{\lambda}(a^\dagger + a). \tag{3.6}$$

$$p_x = \frac{\mathrm{i}}{2\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}, \tag{3.7}$$

$$p_y = \frac{1}{2\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}, \tag{3.8}$$

$$x = \frac{\lambda}{2}\{(a^\dagger + a) + (b^\dagger + b)\}, \tag{3.9}$$

$$y = -\frac{\mathrm{i}\lambda}{2}\{(a^\dagger - a) - (b^\dagger - b)\}, \tag{3.10}$$

where $\omega = \dfrac{eB}{m}$ is the cyclotron frequency and $\lambda = \sqrt{\dfrac{2}{eB}}$. The derivation of this expression is given in Appendix D.1

3.2 Estimation of the position

3.2.1 Setup of the unitary transformations and the reference states

Setup of the unitary transformations

In order to create the unitary operators for the purpose of the position estimation, we have two choices. One is the transformation generated by the mechanical momenta, π_x and π_y. The other is that generated by canonical momenta, p_x and p_y. Below, we see that these two transformations can be used for the position estimation. Hereafter, we call the models generated by π_x and π_y and by p_x and p_y as Model 1 and Model 2, respectively.

Model 1

First, we consider the unitary transformation with the generators, π_x and π_y. The expectation value of x with respect to ρ_θ, $\langle x\rangle_\theta$ is $\langle x\rangle_\theta = \mathrm{tr}\,[\rho_\theta x]$. Therefore,

$$\begin{aligned}\langle x\rangle_\theta &= \mathrm{tr}\,[\rho_\theta x]\\ &= \mathrm{tr}\,[\mathrm{e}^{-\mathrm{i}\theta^1\pi_x}\rho_0\,\mathrm{e}^{\mathrm{i}\theta^1\pi_x}\,x]\\ &= \mathrm{tr}\,[\mathrm{e}^{\mathrm{i}\theta^1\pi_x}\,x\,\mathrm{e}^{-\mathrm{i}\theta^1\pi_x}\rho_0]\end{aligned}$$

By the definition of π_x and π_y, $\mathrm{e}^{\mathrm{i}\theta^1\pi_x}\, x\, \mathrm{e}^{-\mathrm{i}\theta^1\pi_x}$ is expressed as

$$\mathrm{e}^{\mathrm{i}\theta^1\pi_x} x \mathrm{e}^{-\mathrm{i}\theta^1\pi_x} = \mathrm{e}^{\mathrm{i}\theta^1(p_x - \frac{eB}{2})y} x\, \mathrm{e}^{-\mathrm{i}\theta^1(p_x - \frac{eB}{2})y}$$

Since $[p_x,\ y] = 0$ and $[x,\ y] = 0$, we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta^1\pi_x} x \mathrm{e}^{-\mathrm{i}\theta^1\pi_x} &= \mathrm{e}^{\mathrm{i}\theta^1 p_x} \mathrm{e}^{-\mathrm{i}\theta^1 \frac{eB}{2}y} x\, \mathrm{e}^{-\mathrm{i}\theta^1 p_x} \mathrm{e}^{\mathrm{i}\theta^1 \frac{eB}{2}y} \\ &= \mathrm{e}^{\mathrm{i}\theta^1 p_x} x\, \mathrm{e}^{-\mathrm{i}\theta^1 p_x}\end{aligned}$$

From $[x,\ p_x] = \mathrm{i}$ and from Baker-Campbell-Hausdroff formula,

$$\mathrm{e}^B A \mathrm{e}^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots\ , \tag{3.11}$$

we have

$$\mathrm{e}^{\mathrm{i}\theta^1 p_x} x\, \mathrm{e}^{-\mathrm{i}\theta^1 p_x} = x + [\mathrm{i}\theta^1 p_x,\ x] = x - \mathrm{i}\theta^1\,(\mathrm{i}) = x + \theta^1 \tag{3.12}$$

Therefore,

$$\begin{aligned}\langle x\rangle_\theta &= \mathrm{tr}\,[\mathrm{e}^{\mathrm{i}\theta^1\pi_x}\, x\, \mathrm{e}^{-\mathrm{i}\theta^1\pi_x}\rho_0] \\ &= \mathrm{tr}\,[(x + \theta)\rho_0] \\ &= \langle x\rangle_0 + \theta\end{aligned}$$

We can obtain θ by measuring x to get $\langle x\rangle_\theta - \langle x\rangle_0$.

We can show that the same is true for y. Therefore, for the estimation of the position x and y, we can use

$$\rho_\theta^\pi = \mathrm{e}^{-\mathrm{i}\theta^1\pi_x} \mathrm{e}^{-\mathrm{i}\theta^2\pi_y} \rho_0 \mathrm{e}^{\mathrm{i}\theta^2\pi_y} \mathrm{e}^{\mathrm{i}\theta^1\pi_x}. \tag{3.13}$$

Model 2

Next, we consider the unitary transformation with the generators, p_x and p_y.

$$\begin{aligned}\langle x\rangle_\theta &= \mathrm{tr}\,[\rho_\theta x] \\ &= \mathrm{tr}\,[\mathrm{e}^{-\mathrm{i}\theta^1 p_x}\rho_0\, \mathrm{e}^{\mathrm{i}\theta^1 p_x}\, x] \\ &= \mathrm{tr}\,[\mathrm{e}^{\mathrm{i}\theta^1 p_x}\, x\, \mathrm{e}^{-\mathrm{i}\theta^1 p_x}\rho_0]\end{aligned}$$

From (3.12), the unitary transformation $\mathrm{e}^{\mathrm{i}\theta^1 p_x} x \mathrm{e}^{-\mathrm{i}\theta^1 p_x}$ is

$$\mathrm{e}^{\mathrm{i}\theta^1 p_x} x\, \mathrm{e}^{-\mathrm{i}\theta^1 p_x} = x + \theta^1. \tag{3.14}$$

Then, we have

$$\langle x \rangle_\theta = \langle x \rangle_0 + \theta.$$

We can show that the same is true for y. Therefore,

$$\rho_\theta^p = \mathrm{e}^{-\mathrm{i}\theta^1 p_x} \mathrm{e}^{-\mathrm{i}\theta^2 p_y} \rho_0 \mathrm{e}^{\mathrm{i}\theta^2 p_y} \mathrm{e}^{\mathrm{i}\theta^1 p_x}. \tag{3.15}$$

Comparison between Model 1 and Model 2 in respect to the estimation efficiency

From (3.5) (3.6) (3.7) (3.8) (3.9) and (3.10), π_x, π_y, p_x, and p_y can be expressed by a, $a^\dagger$ and b, $b^\dagger$. By comparing the coefficients in π_x and p_x, we find that the coefficient in p_x is $\frac{1}{2}$ of that in π_x. The same is true for π_y and p_y. As can be seen in the section 3.2.5, this difference in the coefficients makes Model 1 better than Model 2 in terms of the quantum estimation. With the same changes in θ^1 and θ^2, Model 1 moves twice as much as Model 2 does. Therefore, we can say that the estimation efficiency is better for Model 1.

Setup of the reference states

Here are the definition of the four reference states. They are a tensor product of two pure states, a tensor product of pure state and mixed state, and a tensor product of mixed state and mixed state. By adding more noisy state, mixed state

Reference state 0 : $\rho_0^{(0)} = |0\rangle_a \langle 0| \otimes |0\rangle_b \langle 0|$

Reference state 1 : $\rho_0^{(1)} = \rho_{0,a} \otimes |0\rangle_b \langle 0|$

Reference state 2 : $\rho_0^{(2)} = \rho_{0,a} \otimes \rho_{0,b}$ (Model 1 : $\kappa \to 0$, Model 2 : $\kappa_b \to 0$ after Fisher information matrices are evaluated.)

Reference state 3 : $\rho_0^{(3)} = \rho_{0,a} \otimes \rho_{0,b}$

where

$$\rho_{0,a} = Z_a^{-1} \sum_{n=0}^{\infty} \mathrm{e}^{-\beta_a n\omega} |n\rangle_a \langle n|$$

$$\rho_{0,b} = Z_b^{-1} \sum_{n=0}^{\infty} \mathrm{e}^{-\beta_b n\omega} |n\rangle_b \langle n|$$

In the following, we focus on the reference state 3, first.

3.2.2 Two-parameter unitary model generated by mechanical momenta π_x and π_y : Model 1

As discussed above, we estimate the position of the electron (x, y) by using the transformation (3.13), i.e.,

$$\rho_\theta^\pi = \mathrm{e}^{-\mathrm{i}\theta^1\pi_x}\mathrm{e}^{-\mathrm{i}\theta^2\pi_y}\rho_0\mathrm{e}^{\mathrm{i}\theta^2\pi_y}\mathrm{e}^{\mathrm{i}\theta^1\pi_x},$$

where ρ_0 is the reference state. Since $\rho_{0,\,b}$ does not play any role in Model 1, we can omit $\rho_{0,\,b}$.

By using the relationships (3.5) and (3.6), we can see that the unitary transformations above depend on $a^\dagger$ and a only.

As the reference state ρ_0, we first examine when the reference state is the mixed state, the thermal state. The thermal state is

$$\rho_0 = Z_\beta^{-1}\mathrm{e}^{-\beta H},$$

where $Z_\beta = \mathrm{tr}\,[\mathrm{e}^{-\beta H}] = \dfrac{\mathrm{e}^{-\frac{1}{2}\beta\omega}}{1-\mathrm{e}^{-\beta\omega}}$ is the partition function. ρ_0 is also expressed in terms of the coherent state as

$$\rho_0 = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z|^2}{2\kappa^2}}\,|z\rangle\,\langle z|\,d^2z,$$

where $2\kappa^2 = \dfrac{\mathrm{e}^{-\beta\omega}}{1-\mathrm{e}^{-\beta\omega}}$ is a parameter related to the temperature, and $|z\rangle$ is the coherent state defined by the annihilation operator, a as $a\,|z\rangle = z\,|z\rangle$. The equivalence between the thermal state and the Gaussian state is given in Appendix D.3.

With the observations above, we confirm that the statistical model (3.15) is identical to the well-known Gaussian shift model [3, 2]. For this model, it is known that RLD Cramer-Rao inequality gives an achievable bound [3, 4]. Since the model (3.15) is a unitary model, $G^R(\theta)$ does not depend on θ, i.e., $G^R(\theta) = G^R(0) =: G^R$. The inverse of the RLD Fisher information $(G^R)^{-1}$ for the model is

$$(G^R)^{-1} = \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa^2 \end{pmatrix}. \tag{3.16}$$

The calculation of Z matrix, RLD and SLD Fisher information matrices are given in Appedix

D.5. Z matrix, $Z(\theta) = [Z_\theta^{ij}]$ is defined by

$$\frac{\partial\rho_\theta}{\partial\theta^i} = \frac{1}{2}(L^S_{\theta,i}\rho_\theta + \rho_\theta L^S_{\theta,i}) \tag{3.17}$$

$$L_0^{S,i} = \sum_j g^{S,ji} L^S_{0,j}, \tag{3.18}$$

$$Z_0^{ij} = \mathrm{tr}[\rho_0 L_0^{S,i}(L_0^{S,j})^\dagger]. \tag{3.19}$$

The RLD Cramer-Rao inequality for any unbiased estimators is

$$V_\theta \geq (G^R)^{-1}, \tag{3.20}$$

where $V_\theta = [V_{\theta,ij}]$ is the mean square error matrix. From the RLD Cramer-Rao inequality and (3.16), we have the following inequality.

$$\{V_{\theta,11} - \frac{\lambda^2}{4}(1+4\kappa^2)\}\{V_{\theta,22} - \frac{\lambda^2}{4}(1+4\kappa^2)\} \geq \frac{\lambda^4}{16}. \tag{3.21}$$

The derivation of this inequality is given in Appendix D.6.4.

As is the case with the RLD Fisher information, the SLD Fisher information, $G^S(\theta)$ does not depend on θ, i.e.,$(G^S(\theta))^{-1} = (G^S(0))^{-1} =: (G^S)^{-1}$, because the model (3.15) is a unitary model. The inverse of the SLD Fisher information, $(G^S)^{-1}$ is

$$(G^S)^{-1} = \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & 0 \\ 0 & 1+4\kappa^2 \end{pmatrix} = \mathrm{Re}\,[(G^R)^{-1}], \tag{3.22}$$

where $\mathrm{Re}\,A = (A + A^*)/2$ denotes the real part of a matrix A.

$\tilde{G^S}(\theta)$ defined by $[\tilde{G^S}(\theta)]_{ij} = \mathrm{tr}\,[\rho_0\, L^S_{0\,j} L^S_{0\,i}]$ is

$$\tilde{G^S}(\theta) = \frac{4}{\lambda^2(1+4\kappa^2)}\begin{pmatrix} 1 & \frac{\mathrm{i}}{1+4\kappa^2} \\ -\frac{\mathrm{i}}{1+4\kappa^2} & 1 \end{pmatrix}$$

Z matrix is

$$\begin{aligned} Z(\theta) &= (G^S)^{-1}(\tilde{G^S})(G^S)^{-1} \\ &= \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa^2 \end{pmatrix} \\ \therefore Z(\theta) &= (G^R)^{-1} \end{aligned}$$

From $Z(\theta) = (G^R)^{-1}$, it turns out that this model is a D-invariant model introduced by Holevo [2]. (3.21) gives an achievable bound. [8]

The SLD Cramer-Rao inequality, $V_\theta \geq (G^S)^{-1}$ gives

$$V_{\theta,11} \geq \frac{\lambda^2}{4}(1 + 4\kappa^2)$$

$$V_{\theta,22} \geq \frac{\lambda^2}{4}(1 + 4\kappa^2)$$

Figure 3.1 shows the RLD Cramer-Rao bound (3.21) and the SLD Cramer-Rao bound above for the temperature parameter $4\kappa^2 = 4$. The shadowed region is the uncertainty relation given by the RLD Cramer-Rao bound. As shown in Figure 3.1, the RLD Cramer-Rao bound is the tight, because this model is D-invariant. In fact, the RLD Cramer-Rao bound gives an achievable bound. [10] Figure 3.2 shows the Cramer-Rao bounds for the different $4\kappa^2$, $4\kappa^2 = 2$, respectively. The SLD and RLD Cramer-Rao bounds move away from the origin= $(0, 0)$ as $4\kappa^2$ increases. This makes sense, because the increase in $4\kappa^2$ means the decrease in $\beta = (kT)^{-1}$ from $2\kappa^2 = \dfrac{1}{e^{\beta\omega} - 1}$.

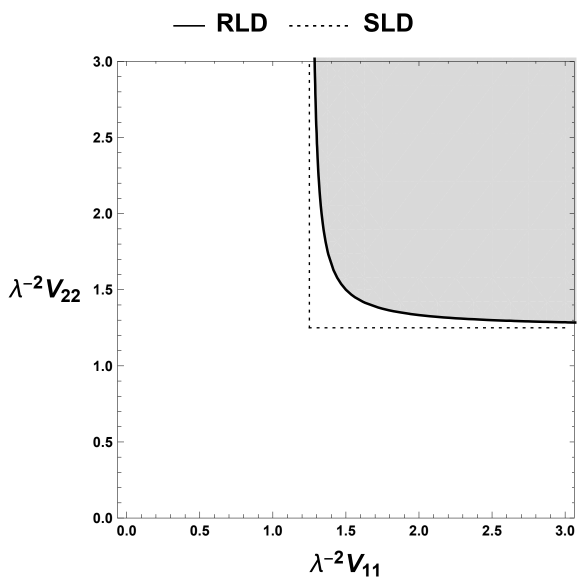

Figure 3.1: Uncertainty relation based on the RLD and SLD Cramer-Rao bounds for $4\kappa^2 = 4$. The shadowed region is the uncertainty relation given by the RLD Cramer-Rao bound.

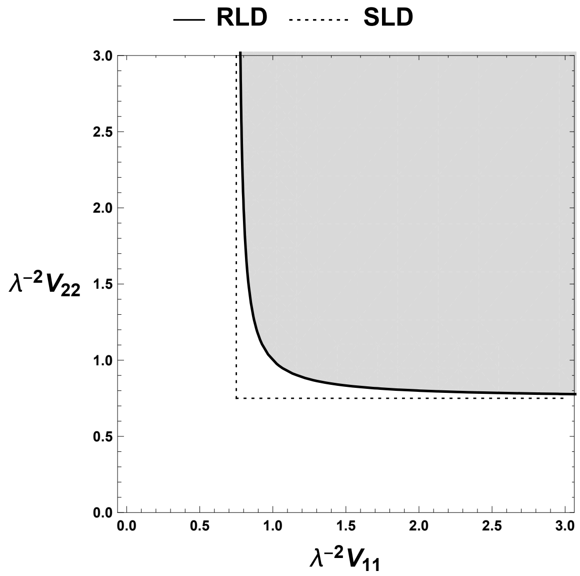

Figure 3.2: Uncertainty relation based on the RLD and SLD Cramer-Rao bounds for $4\kappa^2 = 2$. The shadowed region is the uncertainty relation given by the RLD Cramer-Rao bound.

3.2.3 Two-parameter unitary model generated by canonical momenta p_x and p_y : Model 2 with the reference state 3

Two-dimensional electron state with a constant angular momentum

In the next section, we use an alternative set of parameters to estimate the position of electron (x, y) by the parameter estimation of $\theta = (\theta^1, \theta^2)$ in the two-parameter unitary model generated by p_x and p_y. We first set up the reference state for our state of interest, one electron in a uniform magnetic field $\vec{B}$ with a constant $\langle l_z \rangle$, where $\langle l_z \rangle$ is the expectation value of angular momentum l_z. Therefore, we take the reference state ρ_0 as

$$\rho_0 = Z_{\beta,\mu}^{-1} \mathrm{e}^{-\beta H + \mu l_z},$$

where

$$Z_{\beta,\mu} = \mathrm{tr}\,[\mathrm{e}^{-\beta H + \mu l_z}].$$

The parameter μ is the chemical potential, or Lagrange multiplier. We regard $\langle l_z \rangle$ as one of the control parameters we can give.

Since the two sets of operators, a, $a^\dagger$ and b, $b^\dagger$ act on the two different Hilbert spaces and since we have the relations (3.3) and (3.4), we can define the reference state ρ_0 as

$$\rho_0 = \rho_{0,a} \otimes \rho_{0,b} \tag{3.23}$$

where

$$\rho_{0,a} = \frac{1}{2\pi\kappa_a^2} \int e^{-\frac{|z|^2}{2\kappa_a^2}} |z\rangle_{a\,a}\langle z| d^2 z, \tag{3.24}$$

$$\rho_{0,b} = \frac{1}{2\pi\kappa_b^2} \int e^{-\frac{|z|^2}{2\kappa_b^2}} |z\rangle_{b\,b}\langle z| d^2 z, \tag{3.25}$$

$$2\kappa_a^2 = \frac{\mathrm{e}^{-(\beta\omega+\mu)}}{1-\mathrm{e}^{-(\beta\omega+\mu)}}, \tag{3.26}$$

$$2\kappa_b^2 = \frac{\mathrm{e}^{\mu}}{1-\mathrm{e}^{\mu}}, \tag{3.27}$$

$$a\,|z\rangle_a = z\,|z\rangle_a\,, \tag{3.28}$$

$$b\,|z\rangle_b = z\,|z\rangle_b\,. \tag{3.29}$$

The expectation value of l_z with respect to ρ_0, $\langle l_z\rangle$ is

$$\langle l_z\rangle = 2\kappa_a^2 - 2\kappa_b^2. \tag{3.30}$$

From (3.26), (3.27), and (3.30), we have

$$t(2\kappa_a^2)^2 + (t\langle l_z\rangle - 2)2\kappa_a^2 - 1 - \langle l_z\rangle = 0. \tag{3.31}$$

where $t = \mathrm{e}^{\beta\omega} - 1$. The proper solution is,

$$2\kappa_a^2 = \frac{1}{t} - \frac{\langle l_z\rangle}{2} + \sqrt{\langle l_z\rangle^2 + \frac{4(t+1)}{t^2}}, \tag{3.32}$$

$$2\kappa_b^2 = \frac{1}{t} + \frac{\langle l_z\rangle}{2} + \sqrt{\langle l_z\rangle^2 + \frac{4(t+1)}{t^2}}. \tag{3.33}$$

Figure 3.3 shows μ as a function of $\langle l_z\rangle$ at $\beta\omega = 0.1$, 1, and 5 from top to bottom. μ as a function of $\langle l_z\rangle$ becomes closer to a step function as $\beta\omega$ increases, i.e., the temperature becomes lower. Let μ^* be the chemical potential at $\kappa_a = \kappa_b$ ($\langle l_z\rangle = 0$). From (3.31), we obtain μ^* as

$$\mu^* = -\frac{\beta\omega}{2}. \tag{3.34}$$

Next, with using (D.6.7), (3.9), (3.10), and (3.23), Δx and Δy with respect to the state ρ_0 are calculated as

$$(\Delta x)^2 = (\Delta y)^2 = \frac{\lambda^2}{2}(1 + 2\kappa_a^2 + 2\kappa_b^2). \tag{3.35}$$

$$\therefore\ (\Delta x)(\Delta y) = \frac{\lambda^2}{2}(1 + 2\kappa_a^2 + 2\kappa_b^2). \tag{3.36}$$

The detailed calculation is given in Appendix D.6.7.

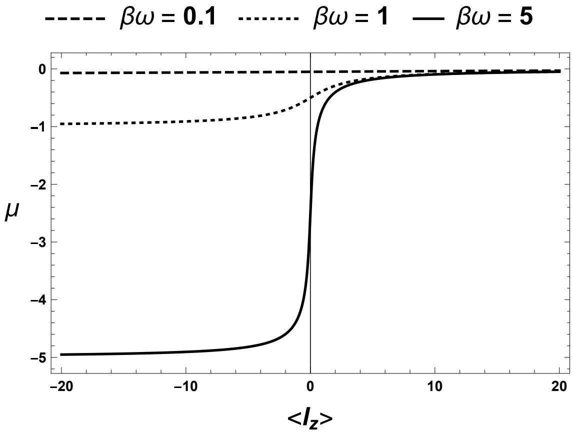

Figure 3.3: The chemical potential μ as a function of $\langle l_z \rangle$.

Unitary model : Model 2

For Model 2, we use the two-parameter unitary model generated by p_x and p_y. As in (3.15), the transformation for Model 2 is

$$\rho_\theta^p = \mathrm{e}^{-\mathrm{i}\theta^1 p_x}\mathrm{e}^{-\mathrm{i}\theta^2 p_y}\rho_0\mathrm{e}^{\mathrm{i}\theta^2 p_y}\mathrm{e}^{\mathrm{i}\theta^1 p_x} \tag{3.37}$$

With using (3.7) and (3.8), $\mathrm{e}^{-\mathrm{i}p_x\theta^1}\mathrm{e}^{-\mathrm{i}p_y\theta^2}$ is

$$\mathrm{e}^{-\mathrm{i}p_x\theta^1}\mathrm{e}^{-\mathrm{i}p_y\theta^2} = \mathrm{e}^{\xi a^\dagger - \xi^* a}\mathrm{e}^{\xi^* b^\dagger - \xi b}, \tag{3.38}$$

where $\xi = \dfrac{1}{2\lambda}(\theta^1 - \mathrm{i}\theta^2)$. Appendix D.2 gives the detailed explanation.

RLD Fisher information

The RLD Fisher information is

$$G^R = G_a^R + G_b^R \tag{3.39}$$

where

$$G_a^R = \frac{1}{4\lambda^2}\begin{pmatrix} \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} & \frac{-\mathrm{i}}{2\kappa_a^2(1+2\kappa_a^2)} \\ \frac{\mathrm{i}}{2\kappa_a^2(1+2\kappa_a^2)} & \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} \end{pmatrix},$$

$$G_b^R = \frac{1}{4\lambda^2}\begin{pmatrix} \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} & \frac{\mathrm{i}}{2\kappa_b^2(1+2\kappa_b^2)} \\ \frac{-\mathrm{i}}{2\kappa_b^2(1+2\kappa_b^2)} & \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} \end{pmatrix}.$$

Its inverse is

$$(G^R)^{-1} = \frac{\lambda^2}{1+2\kappa_a^2+2\kappa_b^2}\begin{pmatrix} 2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 & \mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) \\ -\mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) & 2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 \end{pmatrix} \tag{3.40}$$

From the RLD Cramer-Rao inequality, $V_\theta \geq (G^R(\theta))^{-1}$,

$$(V_{\theta,11} - g^{R,11})(V_{\theta,22} - g^{R,11}) \geq \lambda^4 (\frac{2\kappa_a^2 - 2\kappa_b^2}{1+2\kappa_a^2+2\kappa_b^2})^2 \tag{3.41}$$

where

$$g^{R,11} = [(G^R)^{-1}]_{11} = \lambda^2 \frac{2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2}{1+2\kappa_a^2+2\kappa_b^2}. \tag{3.42}$$

SLD Fisher information

The SLD Fisher information is

$$G^S = \lambda^{-2}\begin{pmatrix} \frac{1}{1+4\kappa_a^2}+\frac{1}{1+4\kappa_b^2} & 0 \\ 0 & \frac{1}{1+4\kappa_a^2}+\frac{1}{1+4\kappa_b^2}. \end{pmatrix} \tag{3.43}$$

Its inverse is

$$(G^S)^{-1} = \lambda^2 \frac{\frac{1}{2}+2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2}{1+2\kappa_a^2+2\kappa_b^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.44}$$

$$\tilde{G^S} = \frac{1}{\lambda^2}\frac{2+4\kappa_a^2+4\kappa_b^2}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2}\begin{pmatrix} (1+4\kappa_a^2)(1+4\kappa_b^2) & \mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) \\ -\mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) & (1+4\kappa_a^2)(1+4\kappa_b^2) \end{pmatrix} \tag{3.45}$$

Then, $Z(\theta) = (G^S)^{-1}\tilde{G^S}(G^S)^{-1}$ is

$$Z(\theta) = \frac{\lambda^2}{1+2\kappa_a^2+2\kappa_b^2}\begin{pmatrix} \frac{1}{2}+2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 & \mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) \\ -\mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) & \frac{1}{2}+2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 \end{pmatrix} \tag{3.46}$$

Since $Z(\theta) \neq (G_\theta^R)^{-1}$, Model 2 is not D-invariant. [8]

From the Cramer-Rao inequality, $V_\theta \geq (G^S)^{-1}$,

$$V_{\theta,11} \geq g^{S,11},\ V_{\theta,22} \geq g^{S,22} = g^{S,11}. \tag{3.47}$$

where

$$g^{S,ij} = [(G^S)^{-1}]_{ij}. \tag{3.48}$$

The derivation of the SLD, the RLD Fisher information matrices, and Z matrix as well is given in Appendix D.6.

Uncertainty relation

The relation between $g^{S,11}$ and $g^{R,11}$ is

$$g^{S,11} = g^{R,11} + \Delta g, \tag{3.49}$$

where

$$\Delta g = \frac{\lambda^2}{2}\frac{1}{1+2\kappa_a^2+2\kappa_b^2} > 0. \tag{3.50}$$

$$\therefore g^{S,11} > g^{R,11} \tag{3.51}$$

Therefore, both the RLD and the SLD Cramer-Rao inequalities need to be incorporated to the uncertainty relation in general. Below we look into this observation in detail.

First, we find out the condition for $(G^S)^{-1} - (G^R)^{-1}$ being positive. From (3.44), (3.40), and (3.50), we obtain

$$(G^S)^{-1} - (G^R)^{-1} = \Delta g \begin{pmatrix} 1 & 2\mathrm{i}\,\langle l_z\rangle \\ -2\mathrm{i}\,\langle l_z\rangle & 1 \end{pmatrix}. \tag{3.52}$$

Therefore, $(G^S)^{-1} \geq (G^R)^{-1}$ is true if and only if $|\langle l_z\rangle| \leq \frac{1}{2}$. Therefore, if and only if $|\langle l_z\rangle| \leq \frac{1}{2}$ holds, the SLD Cramer-Rao bound (3.47) defines a tighter bound.

In the other case, $|\langle l_z\rangle| > \frac{1}{2}$, however, there is no ordering between the RLD and SLD Fisher information matrices in terms of the matrix inequality. This means that both inequalities provide the uncertainty relation. Figure 3.4, 3.5, and 3.6 shows examples of the bound given by the current analysis for $\langle l_z\rangle = 1, \dfrac{1}{2}$, and 0.1 respectively. The upper bounds in the figures are given by Z matrix, because it is known that the Z matrix provides the upper bound. [10]

Since $g^{S,11} > g^{R,11}$, the RLD and SLD Cramer-Rao bounds have two intersection points. Let the position of one of the intersection points be $(V_{11}^{R-S}, g^{S,11})$ which is marked as the dot in Figure 3.4. If $|\langle l_z\rangle| > \frac{1}{2}$ holds (Figure 3.4), the bound is defined by both of the RLD and the SLD Cramer-Rao bounds. The RLD Cramer-Rao bound defines the bound in the region, $g^{S,11} < \lambda^{-2}V_{11} < V_{11}^{R-S}$ and the SLD bound defines at $\lambda^{-2}V_{11} = g^{S,11}$ and at $\lambda^{-2}V_{22} = g^{S,11}$. When $|\langle l_z\rangle| = \frac{1}{2}$ (Figure 3.5), the RLD Cramer-Rao bound touches at the corner of the SLD bound and when $|\langle l_z\rangle| = 0.1 < \frac{1}{2}$, the RLD Cramer-Rao bound and the SLD Cramer-Rao bound no longer have intersections.

We define ΔV^{R-S} by $\Delta V^{R-S} = V_{11}^{R-S} - g^{S,11}$ (Figure 3.5). Then, ΔV^{R-S} is

$$\Delta V^{R-S} = \Delta g(4\langle l_z\rangle^2 - 1). \tag{3.53}$$

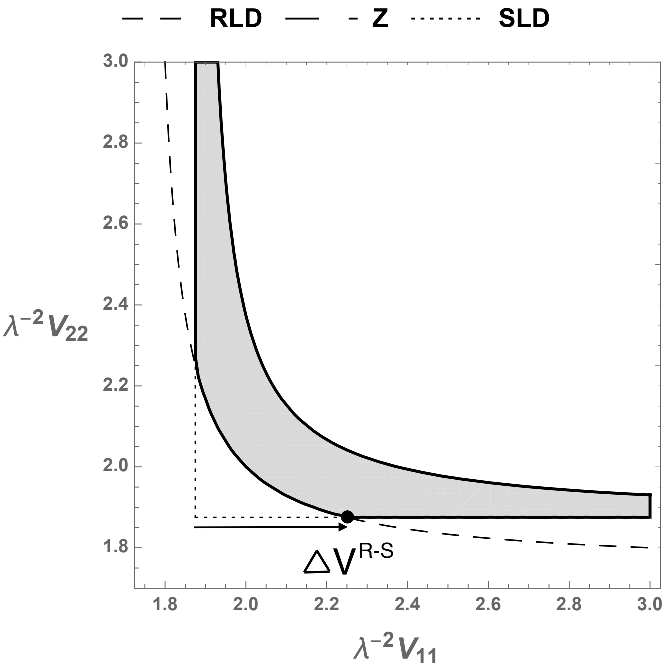

Figure 3.4: The uncertainty relation based on the SLD Cramer-Rao bound, RLD Cramer-Rao bound, and Z matrix for $2\kappa_a^2 = 2, 2\kappa_b^2 = 1, \langle l_z \rangle = 1$. Z matrix defines the upper bound. $\lambda^{-2}V_{11} < \Delta V^{R-S}$, RLD bound defines the lower bound, because RLD Cramer-Rao bound is above SLD Cramer-Rao bound. Otherwise, SLD Cramer-Rao bound defines the lower bound, because SLD Cramer-Rao bound is above RLD Cramer-Rao bound.

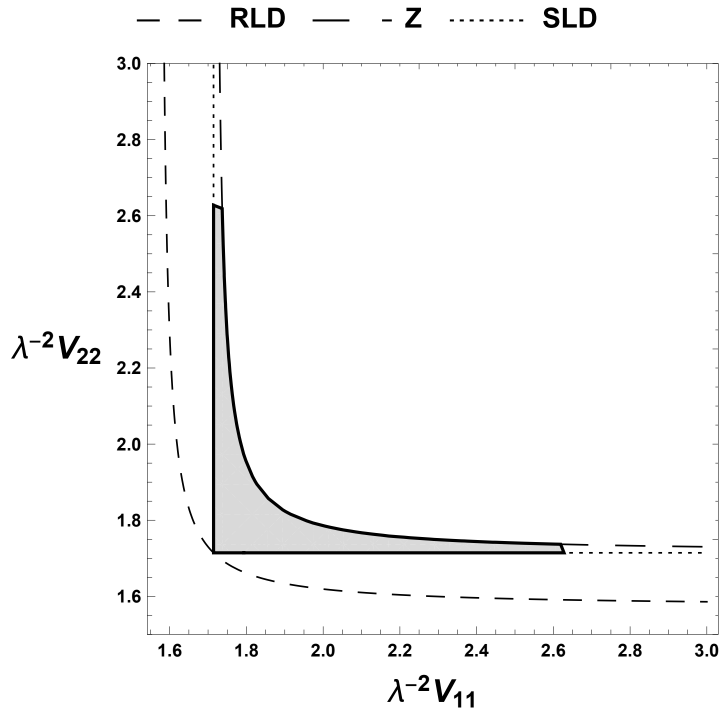

Figure 3.5: The uncertainty relation based on the SLD Cramer-Rao bound, RLD Cramer-Rao bound, and Z matrix for $2\kappa_a^2 = 1.5$, $2\kappa_b^2 = 1$, $\langle l_z \rangle = \frac{1}{2}$. RLD Cramer-Rao bound touches at the corner of the SLD Cramer-Rao bound, because $\langle l_z \rangle = \frac{1}{2}$.

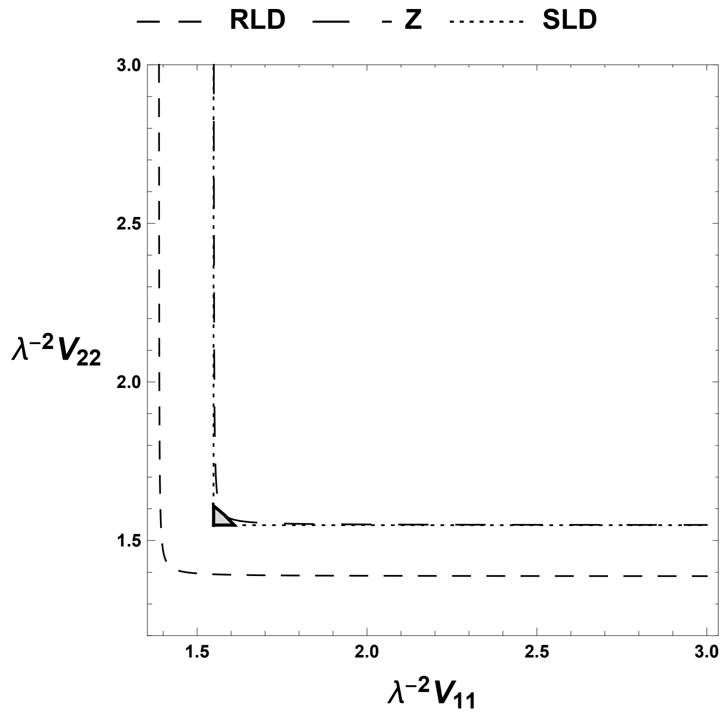

Figure 3.6: The uncertainty relation based on the SLD bound, RLD bound, and Z matrix for $2\kappa_a^2 = 1.1$, $2\kappa_b^2 = 1$ $\langle l_z \rangle = 0.1 < \frac{1}{2}$. RLD bound is lower than the SLD bound, because $\langle l_z \rangle < \frac{1}{2}$.

Figure 3.7 shows ΔV^{R-S} as a function of $\langle l_z \rangle$ at three different $\beta\omega$'s. When $|\langle l_z \rangle| < \frac{1}{2}$, ΔV^{R-S} is

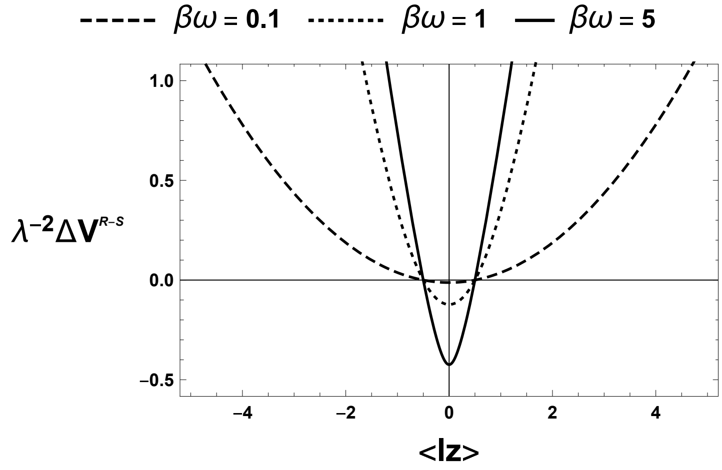

Figure 3.7: ΔV^{R-S} as a function of $\langle l_z \rangle$.

negative (3.53), i.e., the RLD Cramer-Rao bound stays always lower than the SLD Cramer-Rao bound. This is consistent with $(G^S)^{-1} \geq (G^R)^{-1}$ when $|\langle l_z \rangle| \leq \frac{1}{2}$. With larger $\beta\omega$, i.e., at lower temperature, the possible ranges of V_{11}^{R-S} and V_{22}^{R-S} defined by the RLD Cramer-Rao bound become larger at the same $\langle l_z \rangle$.

Finally, we briefly discuss achievability of the above uncertainty relation. With using the $Z(\theta)$ defined by (3.19), it is known that the SLD Cramer-Rao bound is (asymptotically) achievable if and only if $\mathrm{Im}\, Z(\theta) = 0$ [8]. In our model, this is equivalent to $\langle l_z \rangle = 0$. When $\langle l_z \rangle \neq 0$, neither the RLD Cramer-Rao bound nor SLD Cramer-Rao bound is even asymptotically achievable. This is because this model is not D-invariant. (From Eq. (3.49), we have $Z_0^{11} = g^{S,11} \neq g^{R,11}$.) Therefore, the uncertainty relation in this paper is not tight, except for the special choice of the parameter, $\langle l_z \rangle = 0$.

3.2.4 Analysis with pure state and pure state-Gaussian state combination reference state: Model 1 and 2

In the previous sections, we see that Model 1 is D-invariant. We also see that Model 2 is not D-invariant and has a complicated bound when their reference states are the thermal state. The thermal state is equivalent to the reference state 3 : $\rho_0^{(3)} = \rho_{0,a} \otimes \rho_{0\,b}$ defined in the section 3.2.1. To investigated the bounds of these models more in detail, we use the reference state

0 and 1 also defined in the section 3.2.1 as the reference states. They have pure state instead of combination of mixed states. We expect that the different reference states will give us more insight into the results in the previous sections. The reference state 2 is the same as the reference state 3 except for taking the limit after we obtain $(G^R)^{-1}$, $(G^S)^{-1}$ and Z. The reference 2 includes taking the limit of $\kappa \to 0$ (Model 1) or $\kappa_b \to 0$ (Model 2) after the evaluation of $(G^R)^{-1}$, $(G^S)^{-1}$ and Z with respect to the reference state 2. (We expect $\rho_b \to |0\rangle_b \langle 0|$ when $\kappa_b \to 0$.)

The tables below show the $(G^R)^{-1}$, $(G^S)^{-1}$ and Z for the reference states 0 to 3 and for Model 1 and 2. The derivation of $(G^R)^{-1}$, $(G^S)^{-1}$ and Z is given in Appendix E.

Table 3.1: Model 1

Reference state	G^R	$(G^R)^{-1}$	$(G^S)^{-1}$	Z
$\rho^{(0)}$	N/A	N/A	$\frac{\lambda^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}$
$\rho^{(1)}$		$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & \mathrm{i}\\ -\mathrm{i} & 1+4\kappa^2\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & 0\\ 0 & 1+4\kappa^2\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & \mathrm{i}\\ -\mathrm{i} & 1+4\kappa^2\end{pmatrix}$
$\rho^{(2)}$		$\frac{\lambda^2}{4}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}$
$\rho^{(3)}$		$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & \mathrm{i}\\ -\mathrm{i} & 1+4\kappa^2\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & 0\\ 0 & 1+4\kappa^2\end{pmatrix}$	$\frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & \mathrm{i}\\ -\mathrm{i} & 1+4\kappa^2\end{pmatrix}$

Table 3.2: Model 2

Reference state	G^R	$(G^R)^{-1}$	$(G^S)^{-1}$	Z
$\rho^{(0)}$	N/A	N/A	$\frac{\lambda^2}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{\lambda^2}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
$\rho^{(1)}$	N/A	N/A	$\frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{\lambda^2}{2}\frac{1}{1+2\kappa_a^2}\begin{pmatrix}1+4\kappa_a^2 & -\mathrm{i}\,4\kappa_a^2\\ \mathrm{i}\,4\kappa_a^2 & 1+4\kappa_a^2\end{pmatrix}$
$\rho^{(2)}$ $\kappa_b \to 0$		$\frac{\lambda^2 2\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}$	$\frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{\lambda^2}{2}\frac{1}{1+2\kappa_a^2}\begin{pmatrix}1+4\kappa_a^2 & -\mathrm{i}4\kappa_a^2\\ \mathrm{i}4\kappa_a^2 & 1+4\kappa_a^2\end{pmatrix}$
$\rho^{(3)}$		(3.40)	(3.44)	(3.46)

3.2.5 Discussion

It is worth noting that there is a difference in the coefficients of $(G^S)^{-1}$ and Z for the reference state 0 of Model 1 and Model. The coefficient of Model 1 is $\frac{\lambda^2}{4}$ and that of Model 2 is $\frac{\lambda^2}{2}$. As is given in Appendix D.1, π_x, π_y and p_x, p_y have a difference of factor of $\frac{1}{2}$. Here are the π_x, π_y and p_x, p_y.

$$\begin{aligned}\pi_x &= \frac{1}{\mathrm{i}\lambda}(a - a^\dagger)\\ \pi_y &= \frac{1}{\lambda}(a + a^\dagger)\end{aligned}$$

$$\begin{aligned}p_x &= \frac{1}{2\mathrm{i}\lambda}(b - b^\dagger) + \frac{1}{2\mathrm{i}\lambda}(a - a^\dagger)\\ p_y &= -\frac{1}{2\lambda}(b + b^\dagger) + \frac{1}{2\lambda}(a + a^\dagger)\end{aligned}$$

This difference makes the difference between Model 1 and 2 regarding the coefficients $(G^S)^{-1}$ and Z for the reference 0. Since Model 2 has a larger factor, we can say that Model 2 gives the worse estimation of Quantum estimation point of view.

Reference state 0 $\rho^{(0)}$: Model 1

$(G^S)^{-1} \neq Z$ indicates that $(G^S)^{-1}$ should not be a good bound. We need both $(G^S)^{-1}$ and Z matrix to defined the bound. Figure 3.8 shows the bounds given by $(G^S)^{-1}$ and Z matrix.

Reference state 0 $\rho^{(0)}$: Model 2

In this case, $(G^S)^{-1} = Z$. Therefore, $(G^S)^{-1}$ gives an achievable bound. Figure 3.9 shows the SLD Cramer-Rao bound which is achievable. The reference state 0 for Model 2 corresponds to the result of the reference state 0 $\rho^{(2)}$: Model 2 with the limit of $\kappa_a,\ \kappa_b \to 0$. We also know that $[L_1^S, L_2^S] = 0$ if $\kappa_a = \kappa_b$. Therefore, L_1^S and L_2^S can be diagonalized simultaneously which is why $(G^S)^{-1}$ gives an achievable bound and also makes $(G^S)^{-1} = Z$. The proof of $[L_1^S, L_2^S] = 0$ if $\kappa_a = \kappa_b$ is given in Appendix F. This is completely different with the result of the reference state 0 $\rho^{(0)}$: Model 1 shown in Figure 3.8, because in that case, the generators of the transformation, π_x and π_y do not commute and this case is quantum mechanical.

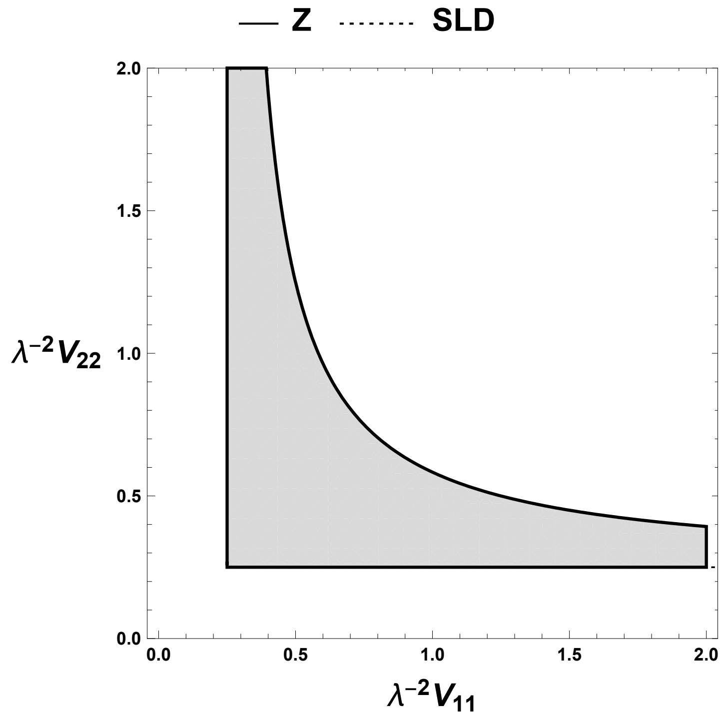

Figure 3.8: SLD and Z matrix bounds for the reference state 0 : Model 1

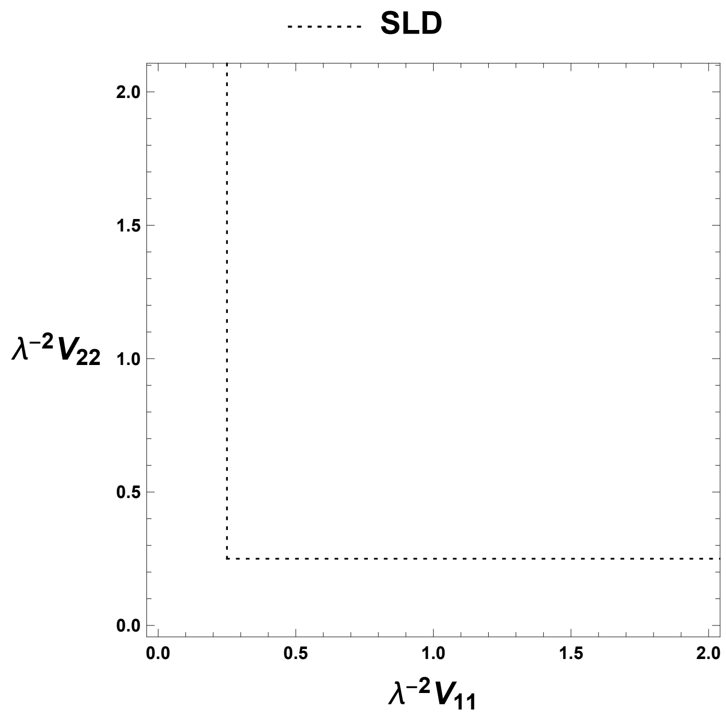

Figure 3.9: SLD bound for the reference state 0 : Model 2. SLD is an achievable bound.

Reference state 1 $\rho^{(1)}$: Model 1

As it has been already mentioned in the previous section, $(G^R)^{-1} = Z$. RLD Cramer-Rao bound is achievable. When $\kappa \to 0$, $(G^S)^{-1}$ and Z of the reference state 1 are the same as those of the reference 0.

Reference state 1 $\rho^{(1)}$: Model 2

Here again, when $\kappa_a \to 0$, $(G^S)^{-1}$ and Z of the reference state 1 are the same as those of the reference 0. SLD Cramer-Rao bound is achievable bound.

Reference state 2 $\rho^{(2)}$: Model 2

The result of the reference state 2 $\rho^{(2)}$: Model 2 is the same as that of the reference state 1 $\rho^{(1)}$: Model 2 except for $(G^R)^{-1}$. Figures 3.10 and 3.11 show the Cramer-Rao bounds for $2\kappa_a^2 =$ 2 and 3, respectively. As seen with the thermal reference state, $(G^R)^{-1}$ does not give effect on the bound when $\langle l_z \rangle = \frac{1}{2}$. Figure 3.11 shows the case for $\langle l_z \rangle = \frac{1}{2}$.

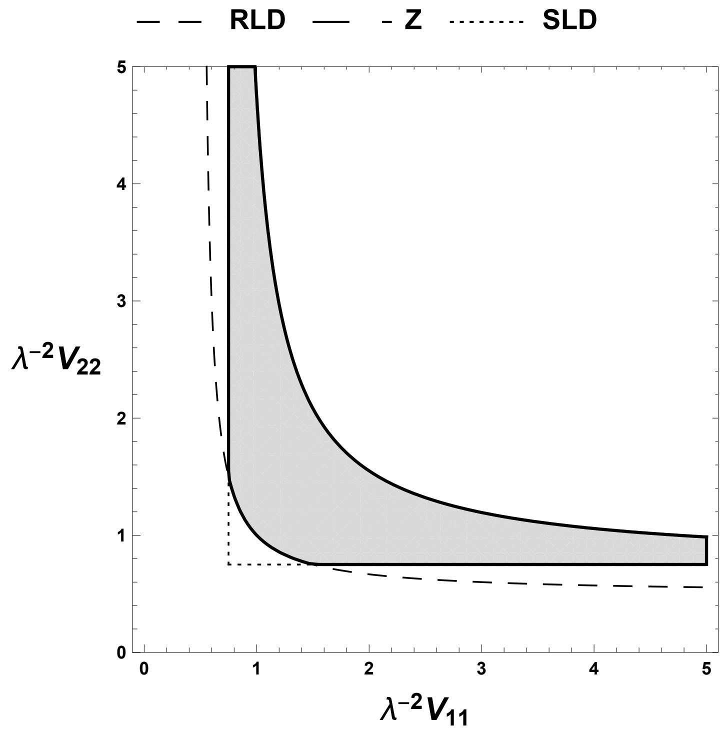

Figure 3.10: Reference state 2 $\rho^{(2)}$: Model 2 $2\kappa_a^2 = 1, \langle l_z \rangle = 1$

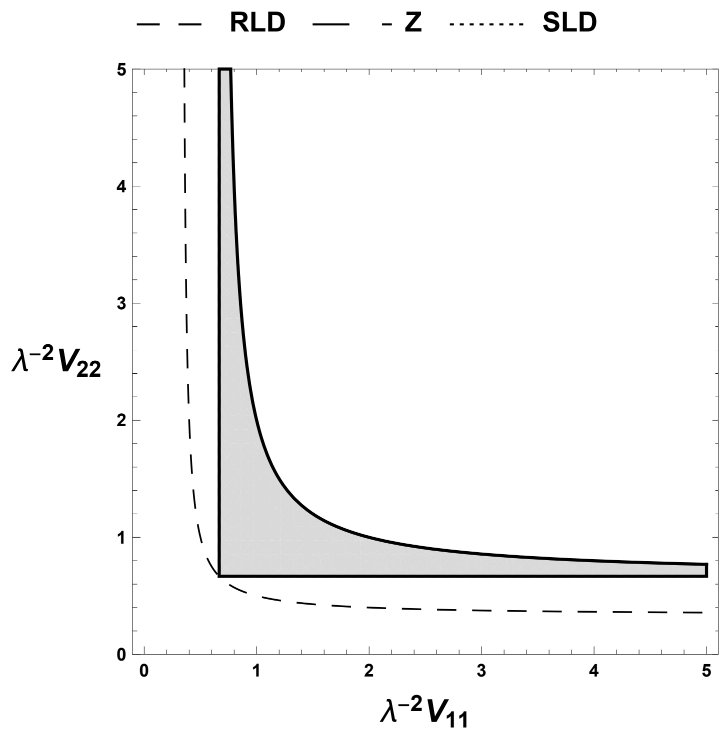

Figure 3.11: Reference state 2 $\rho^{(2)}$: Model 2 $2\kappa_a^2 = 0.5, \langle l_z \rangle = \frac{1}{2}$

Reference state 3 $\rho^{(3)}$: Model 1

As give in the section 3.2.2, this model is a D-invariant model. The RLD Cramer-Rao bound is tight and is achievable, because this is equivalent to the Gaussian shift model.

Reference state 3 $\rho^{(3)}$: Model 2

We assumed that we who make measurement will give a certain $\langle l_z \rangle$, or control $\langle l_z \rangle$. With this assumption, we see the shape of the bound depend on $\langle l_z \rangle$. It is quite interesting that the angular momentum changes the shape of the bound. When $|\langle l_z \rangle| > \frac{1}{2}$, the bound is defined by the $(G^R)^{-1}$, $(G^S)^{-1}$, and Z matrix. (Figure 3.4) When $|\langle l_z \rangle| < \frac{1}{2}$, the RLD bound gives no contribution to defining the bound, because at $|\langle l_z \rangle| = \frac{1}{2}$ (Figure 3.5 and 3.6), the RLD Cramer-Rao bound touches at the corner of the SLD Cramer-Rao bound and because when $|\langle l_z \rangle| = 0.1 < \frac{1}{2}$, it goes off the RLD Cramer-Rao bound and goes further down.

Chapter 4

Summary and outlook

4.1 Summary

We have investigated the uncertainty relation of one electron in a uniform magnetic field by the parameter estimation of $\theta = (\theta^1, \theta^2)$ in the two-parameter unitary models. Two different sets of generators for the unitary transformation are used. One is the set of mechanical momenta, π_x and π_y (Model 1), and the other is the set of canonical momenta, p_x and p_y (Model 2). In the both cases, we got the non-trivial bounds unlike the result of Heisenberg-Robertson uncertainty relation. In the former, the RLD Cramer-Rao bound gives an achievable bound. This is because Model 1 is equivalent to Gaussian shift model.

For Model 2, we have showed that the obtained bound is defined by $(G^R)^{-1}$, $(G^S)^{-1}$, and Z matrix when the angular momentum of the system is fixed and its magnitude exceeds $\frac{1}{2}$. (Figure 3.4)

If we are to execute the experiment for Model 2, we cannot measure at the accuracy that is indicated within the area lower than the shadowed area in the figure 3.4 according to our result. The approach of this kind is helpful for designing an experiment for phenomena with a quantum mechanical nature.

Model 1 When the reference state is the pure state, i.e., the reference state 0, we have $(G^S)^{-1} \neq Z$. This is because π_x and π_y do not commute, therefore the unitary transformations generated by π_x and π_y have the quantum mechanical nature (Figure 3.8). The quantum effect is more prominent than the case of reference 1 below, because the pure state is less noisy than the mixed state.

In the meantime, when the reference state is the mixed state, the reference state 1, Model 1 is equivalent to the Gaussian shift model. We see that the RLD Cramer-Rao inequality gives an achievable bound as usually seen in the Gaussian shift model. This is because only one set of the creation-annihilation operators, a, $a^\dagger$ is the generators of the transformation in Model 1.

Model 2 Model 2 has a quasi-classical nature, because p_x and p_y commute. For the pure state reference state, the reference state 0, the SLD Cramer-Rao bound is achievable. Its bound is simpler than the bound for the reference 0 : Model 1 (Figure 3.9).

The generators of the transformation for Model 2 are two sets of the creation-annihilation operators operators, a, $a^\dagger$ and b, $b^\dagger$.

Although Model 2 has the commutable generators that gives a quasi-classical effect, in the case of the reference state 3, a different situation arises. The reference state 3 is defined as the tensor product of two Gaussian model, i.e.,

$$\rho_0^{(3)} = \rho_{0,a} \otimes \rho_{0,b}$$

The two are not just a tensor product of independent two states, $\rho_{0,a}$ and $\rho_{0,b}$. As given in Appendix (D.2), 3.38 is

$$\mathrm{e}^{-\mathrm{i}p_x\theta^1}\mathrm{e}^{-\mathrm{i}p_y\theta^2} = \mathrm{e}^{\xi a^\dagger - \xi^* a}\mathrm{e}^{\xi^* b^\dagger - \xi b},$$

where $\xi = \dfrac{1}{2\lambda}(\theta^1 - \mathrm{i}\theta^2)$.
The coefficients of a, $a^\dagger$ and b, $b^\dagger$ are not independent. Therefore, the coefficient of z_a in $\rho_{\theta,a}$ and z_b in $\rho_{\theta,b}$ are not independent and their relation is $z_a = z_b^*$. This makes the state $\rho_{0,a}$ and $\rho_{0,b}$ dependent each other and makes the result of the transformation more complicated. The bound shape discontinuously changes at $\langle l_z \rangle = \frac{1}{2}$.

4.2 Outlook

Since Model 2 is not D-invariant, we need to work on Model 2 with Holevo bound. We will further study Model 2 with that perspective.

We also worked on Model 1 with taking relativity into account, i.e, with Hamiltonian based on the Dirac equation. Although we found that the model is expressed as a special case of Jaynes-Cumming model, we cannot get a physically reasonable result. We believe that it is probably not possible to avoid divergence when the thermal state is used as the reference state in the relativistic case. We try simpler reference states to pursue a meaningful result. Our result so far is given in Appendix G.

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Appendix A

Uncertainty relations

A.1 Time - Energy and number - phase uncertainty

From Heisenberg equation $\frac{d\rho_{\theta_0}}{dt} = -\mathrm{i}[H, \rho(t)]$, we can regard the time measurement as the estimation of a parameter θ of the equation $\frac{d\rho_\theta}{d\theta} = -\mathrm{i}[H, \rho_\theta]$. We will derive Time-Energy uncertainty relation with using the assumption above. We denote the variance as $V_\theta(\hat{\theta}) = \langle (H - \langle H \rangle)^2 \rangle$.

Proof [4]

Anti-symmetric logarithmic derivative, or ALD is defined by

$$\frac{d\rho_\theta}{d\theta} = \frac{1}{2}(\rho_\theta L_\theta + L_\theta^\dagger \rho_\theta)$$

If $L_\theta = 2\mathrm{i}(H - C)$, where C is a real constant, then

$$\begin{aligned}
\frac{1}{2}(\rho_\theta L_\theta + L_\theta^\dagger \rho_\theta) &= -\mathrm{i}(\rho_\theta(H - C) + (-H + C)\rho_\theta) \\
&= -\mathrm{i}[H, \rho_\theta] \\
&= \frac{d\rho_\theta}{d\theta} \\
C = \langle H \rangle & \quad \because \mathrm{tr}\,[\rho_\theta L_\theta] = 0
\end{aligned}$$

Therefore

$$L_\theta = 2\mathrm{i}(H - \langle H \rangle_\rho) \tag{A.1}$$

We assume that T is an unbiased estimator, that is $\langle T \rangle_\rho = \theta$

$$\frac{d}{d\theta}\langle T \rangle_\rho = \frac{d}{d\theta}\mathrm{tr}\,[\rho_\theta T] = \mathrm{tr}\,[\frac{d\rho_\theta}{d\theta} T] = \mathrm{tr}\,[\frac{d\rho_\theta}{d\theta}(T - t)] \tag{A.2}$$

When t = constant, $\mathrm{tr}\,[\frac{d\rho_\theta}{d\theta}t] = \frac{d}{d\theta}\mathrm{tr}\,[\rho_\theta t] = \frac{d}{d\theta}t = 0$ was used.

Since T is an unbiased estimator, $\langle T\rangle_\rho = \theta$. $\frac{d}{d\theta}\langle T\rangle_\rho = 1$

Therefore,

$$
\begin{aligned}
\frac{d}{d\theta}\langle T\rangle_\rho &= \mathrm{tr}\,[\frac{d\rho_\theta}{d\theta}(T-t)] \\
&= \mathrm{tr}\,[\frac{1}{2}(\rho_\theta L_\theta + L_\theta^\dagger \rho_\theta)(T-t)] \\
&= \frac{1}{2}\mathrm{tr}\,[\rho_\theta L_\theta (T-t)] + \frac{1}{2}\mathrm{tr}\,[(\rho_\theta L_\theta)^\dagger (T-t)] \\
&= \frac{1}{2}\mathrm{tr}\,[\rho_\theta L_\theta (T-t)] + \frac{1}{2}\mathrm{tr}\,[(T-t)(\rho_\theta L_\theta)^\dagger] \\
&= \frac{1}{2}\mathrm{tr}\,[\rho_\theta L_\theta (T-t)] + \frac{1}{2}\mathrm{tr}\,[(\rho_\theta L_\theta (T-t))^\dagger] \\
&= \frac{1}{2}\mathrm{tr}\,[\rho_\theta L_\theta (T-t)] + \frac{1}{2}(\mathrm{tr}\,[\rho_\theta L_\theta (T-t)])^* \\
&= \mathrm{Re}\,\{\mathrm{tr}\,[(\rho_\theta L_\theta)(T-t)]\} = 1 \\
&= \mathrm{Re}\,\{\langle L_\theta, T-t\rangle_\rho^R\} = 1
\end{aligned}
$$

Cauchy-Schwartz inequality gives

$$
\begin{aligned}
&||L_\theta||_\theta^2\,\langle (T-t)^2\rangle_\rho \geq |\langle L_\theta, T-t\rangle_{\rho_\theta}^R|^2 \geq 1 \\
&\because\ |\langle L_\theta, T-t\rangle_{\rho_\theta}^R|^2 = \mathrm{Re}\,|\langle L_\theta, T-t\rangle_\rho^R|^2 + \mathrm{Im}\,|\langle L_\theta, T-t\rangle_{\rho_\theta}^R|^2 \geq \mathrm{Re}\,|\langle L_\theta, T-t\rangle_{\rho_\theta}^R|^2 = 1
\end{aligned}
$$

From (A.1)

$$
\begin{aligned}
||L_\theta||_\theta^2 &= \mathrm{tr}\,[\rho_\theta L_\theta^\dagger L_\theta] \\
&= \mathrm{tr}\,[\rho_\theta(-2\mathrm{i})(H-\langle H\rangle_\rho)(2\mathrm{i})(H-\langle H\rangle_\rho)] \\
&= 4\mathrm{tr}\,[\rho_\theta(H-\langle H\rangle_\rho)^2] \\
&= 4\mathrm{tr}\,[\rho_\theta(H^2 - 2H\langle H\rangle_\rho + \langle H\rangle_\rho^2] \\
&= 4\mathrm{tr}\,[\rho_\theta(H^2 - \langle H\rangle_\rho^2] \\
&= 4\mathrm{tr}\,[\rho_\theta(H^2 - \langle H\rangle_\rho^2] \\
&= 4\,(\langle H^2\rangle_{\rho_\theta} - \langle H\rangle_\rho^2) \\
\langle (T-t)^2\rangle_\rho &= 4\,(\langle H^2\rangle_{\rho_\theta} - \langle H\rangle_\rho^2)
\end{aligned}
$$

t can be any real number, therefore we set t as $t = \theta$

$$
\begin{aligned}
\langle (T-\theta)^2\rangle_\rho &= V_\theta(\hat{\theta}) \\
V_\theta(\hat{\theta})(\langle H^2\rangle_\rho - \langle H\rangle_\rho^2) &\geq \frac{1}{4}
\end{aligned}
$$

Here we obatin

$$\Delta T \Delta H \geq \frac{1}{2}. \tag{A.3}$$

We can regard this as a uncertainty relation between time and energy.

A.2 Heisenberg-Schrodinger-Robertson uncertainty relation

Let us set the Hilbert space$\mathscr{H} = \boldsymbol{C}^d$, the given obervables are $X, Y \in \mathscr{L}_n(\mathscr{H})$, and the state $\rho \in \mathscr{S}(\mathscr{H})$.

Q-covariance $\Delta_\rho(X, Y)$ is defined by

$$\Delta_\rho(X, Y) = \begin{pmatrix} \langle (X - \langle X \rangle_\rho) \rangle_\rho^2 & \langle (X - \langle X \rangle_\rho)(Y - \langle Y \rangle_\rho) \rangle_\rho \\ \langle (Y - \langle Y \rangle_\rho)(X - \langle X \rangle_\rho) \rangle_\rho & \langle (Y - \langle Y \rangle_\rho) \rangle_\rho^2 \end{pmatrix} \tag{A.4}$$

$$\langle X - \langle X \rangle_\rho) \rangle_\rho^2 = \langle X^2 \rangle_\rho - \langle X \rangle_\rho^2 \tag{A.5}$$

and

$$\begin{aligned}
\langle (X - \langle X \rangle_\rho)(Y - \langle Y \rangle_\rho) \rangle_\rho &= \langle XY - X \langle Y \rangle_\rho - \langle Y \rangle_\rho X + \langle X \rangle_\rho \langle Y \rangle_\rho \rangle_\rho && \text{(A.6)} \\
&= \langle XY \rangle_\rho - \langle X \rangle_\rho \langle Y \rangle_\rho - \langle Y \rangle_\rho \langle X \rangle_\rho + \langle X \rangle_\rho \langle Y \rangle_\rho && \text{(A.7)} \\
&= \langle XY \rangle_\rho - \langle X \rangle_\rho \langle Y \rangle_\rho && \text{(A.8)} \\
\langle (Y - \langle Y \rangle_\rho)(X - \langle X \rangle_\rho) \rangle_\rho &= \langle YX \rangle_\rho - \langle Y \rangle_\rho \langle X \rangle_\rho && \text{(A.9)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta_\rho(X, Y) &= \begin{pmatrix} \langle X^2 \rangle_\rho - \langle X \rangle_\rho^2 & \langle XY \rangle_\rho - \langle X \rangle_\rho \langle Y \rangle_\rho \\ \langle YX \rangle_\rho - \langle Y \rangle_\rho \langle X \rangle_\rho & \langle Y^2 \rangle_\rho - \langle Y \rangle_\rho^2 \end{pmatrix} && \text{(A.10)} \\
&= \begin{pmatrix} \langle X^2 \rangle_\rho & \langle XY \rangle_\rho \\ \langle YX \rangle_\rho & \langle Y^2 \rangle_\rho \end{pmatrix} - \begin{pmatrix} \langle X \rangle_\rho \\ \langle Y \rangle_\rho \end{pmatrix} \begin{pmatrix} \langle X \rangle_\rho \langle Y \rangle_\rho \end{pmatrix}
\end{aligned}$$

where $\langle X \rangle_\rho = \operatorname{tr} \rho X$.

The following three things should be noted.

1. $\langle XY \rangle_\rho = \operatorname{tr} \rho XY \neq \langle YX \rangle_\rho$ in general. ($\langle XY \rangle_\rho \in \boldsymbol{C}$) in general

2. There exist many possibilites for defining quantum covariance.

3. $V_\rho(X) := \langle(X-\langle X\rangle_\rho)\rangle_\rho^2,\ V_\rho(Y) = \langle(Y-\langle Y\rangle_\rho)\rangle_\rho^2$ are the variances of X and Y.

Here, we introduce a Lemma.

Lemma

$$\mathrm{Re}\langle XY\rangle_\rho = \frac{1}{2}\mathrm{tr}\,\rho\{X,Y\} \tag{A.11}$$

$$\mathrm{Im}\langle XY\rangle_\rho = \frac{1}{2\mathrm{i}}\mathrm{tr}\,\rho[X,Y] \tag{A.12}$$

$$\begin{aligned}\because)\ \mathrm{Re}\,\langle XY\rangle_\rho &= \frac{1}{2}(\langle XY\rangle_\rho + (\langle XY\rangle_\rho)^*) = \frac{1}{2}(\mathrm{tr}\,\rho XY + \mathrm{tr}\,YX\rho) = \frac{1}{2}\mathrm{tr}\,\rho\{X,Y\}\\ \mathrm{Im}\,\langle XY\rangle_\rho &= \frac{1}{2\mathrm{i}}(\langle XY\rangle_\rho - (\langle XY\rangle_\rho)^*) = \frac{1}{2\mathrm{i}}(\mathrm{tr}\,\rho XY - \mathrm{tr}\,YX\rho) = \frac{1}{2\mathrm{i}}\mathrm{tr}\,\rho[X,Y]\end{aligned}$$

Then, we come to the theorem as follows.

1. Robertson : $V_\rho(X)V_\rho(Y) \geq \frac{1}{2}\mathrm{tr}\,\rho[X,Y] = |\mathrm{Im}\langle XY\rangle_\rho|^2$

2. Schrodinger : $V_\rho(X)V_\rho(Y) \geq \frac{1}{2}\mathrm{tr}\,\rho[X,Y] = |\mathrm{Re}\langle XY\rangle_\rho - \langle X\rangle_\rho\langle Y\rangle_\rho|^2|\mathrm{Im}\langle XY\rangle_\rho|^2$

Remarks

1. for $\rho = |\psi\rangle\langle\psi|$

$V_\rho(X) = \langle X^2\rangle_\psi - \langle X\rangle_\psi^2,\ \ \mathrm{tr}\,\rho[X,Y] = \langle[X,Y]\rangle_\psi$

2. Obviously, Schrodinger version is stronger than Roberson version.

$\because)\ |\mathrm{Re}\langle XY\rangle_\rho|^2 \geq 0$

Since Schrodinger version is stronger than Roberson version, we prove Schrodinger version.

Proof (Schrodinger)

Let A be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.13}$$

$$A \geq 0 \Rightarrow \det A \geq 0 \Leftrightarrow ad \geq |b|^2 = |c|^2 \tag{A.14}$$

Q-covariance, $\Delta_\rho(X,Y) \geq 0$.

Let $|c)$ be

$$|c) = \begin{pmatrix} c_x \\ c_y \end{pmatrix} \in \mathbb{C}^2 \tag{A.15}$$

$$
\begin{aligned}
(c|\Delta_\rho(X,Y)|c) &= \begin{pmatrix} c_x^* & c_y^* \end{pmatrix} \begin{pmatrix} \langle X^2\rangle_\rho - \langle X\rangle_\rho^2 & \langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho \\ \langle YX\rangle_\rho - \langle Y\rangle_\rho \langle X\rangle_\rho & \langle Y^2\rangle_\rho - \langle Y\rangle_\rho^2 \end{pmatrix} \begin{pmatrix} c_x \\ c_y \end{pmatrix} \\
&= \begin{pmatrix} c_x^*(\langle X^2\rangle_\rho - \langle X\rangle_\rho^2) + c_y^*(\langle YX\rangle_\rho - \langle Y\rangle_\rho \langle X\rangle_\rho) & c_x^*(\langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho) + c_y^*(\langle Y^2\rangle_\rho - \langle Y\rangle_\rho^2) \end{pmatrix} \begin{pmatrix} c_x \\ c_y \end{pmatrix} \\
&= |c_x|^2(\langle X^2\rangle_\rho - \langle X\rangle_\rho^2) + c_x c_y^*(\langle YX\rangle_\rho - \langle Y\rangle_\rho \langle X\rangle_\rho) + c_y c_x^*(\langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho) + |c_y|^2(\langle Y^2\rangle_\rho - \langle Y\rangle_\rho^2)
\end{aligned}
$$

We define C_x, $C_x^\dagger$ and C_x, $C_x^\dagger$ as

$$
\begin{aligned}
C_x &:= c_x X, \;\; C_x^\dagger := c_x^* X \\
C_y &:= c_y Y, \;\; C_y^\dagger := c_y^* Y
\end{aligned}
$$

Then,

$$
\begin{aligned}
(c|\Delta_\rho(X,Y)|c) &= \langle (C_x^\dagger - \langle C_x^\dagger\rangle_\rho)(C_x - \langle C_x\rangle_\rho)\rangle_\rho + \langle (C_x^\dagger - \langle C_x^\dagger\rangle_\rho)(C_y - \langle C_y\rangle_\rho)\rangle_\rho \\
&\quad + \langle (C_y^\dagger - \langle C_y^\dagger\rangle_\rho)(C_x - \langle C_x\rangle_\rho)\rangle_\rho + \langle (C_y^\dagger - \langle C_y^\dagger\rangle_\rho)(C_y - \langle C_y\rangle_\rho)\rangle_\rho \\
&= \langle (C_x^\dagger - \langle C_x^\dagger\rangle_\rho + C_y^\dagger - \langle C_y^\dagger\rangle_\rho)(C_x - \langle C_x\rangle_\rho + C_y - \langle C_y\rangle_\rho)\rangle_\rho \geq 0 \\
&\because \mathrm{tr}\,[A^\dagger A\rho] \geq 0
\end{aligned}
\tag{A.16}
$$

By the circularity of trace, we have

$$
\mathrm{tr}\,[A^\dagger A\rho] = \mathrm{tr}\,[A\rho A^\dagger] \geq 0
$$

$$
\begin{aligned}
\Delta_\rho(X,Y) &\geq 0 \\
&\Rightarrow \det \Delta_\rho(X,Y) \geq 0 \\
&\Leftrightarrow V_\rho(X)V_\rho(Y) \geq |\langle (X - \langle X\rangle_\rho)(Y - \langle Y\rangle_\rho)\rangle_\rho|^2 \\
&= |\langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho|^2 \\
&= |\langle \mathrm{Re}\langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho + \mathrm{i}\,\mathrm{Im}\langle XY\rangle_\rho|^2 \\
&= |\langle \mathrm{Re}\langle XY\rangle_\rho - \langle X\rangle_\rho \langle Y\rangle_\rho|^2 + |\mathrm{Im}\langle XY\rangle_\rho|^2 \\
&= |\langle \frac{1}{2}\mathrm{tr}\,\rho\{X,Y\} - \langle X\rangle_\rho \langle Y\rangle_\rho|^2 + |\frac{1}{2}\mathrm{tr}\,\rho[X,Y]|^2 \\
&= \frac{1}{4}|\langle \mathrm{tr}\,\rho\{X - \langle X\rangle_\rho, Y - \langle Y\rangle_\rho\}\rangle_\rho|^2 + \frac{1}{4}|\mathrm{tr}\,\rho[X,Y]|^2
\end{aligned}
\tag{A.17}
$$

Appendix B

One-parameter estimation : Gaussian model

As a reference state, Gaussian state is chosen.

$$\rho_0 = \frac{1}{2\pi K^2} \int e^{-\frac{|z'-z|^2}{2K^2}} |z'\rangle \langle z'| \, d^2 z' \tag{B.1}$$

$|z\rangle$ is the coherence state defined by

$$A |z\rangle = z |z\rangle \tag{B.2}$$

$$[Q, P] = QP - PQ = \mathrm{i} \tag{B.3}$$

$$A = \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP) \tag{B.4}$$

where is any positive real number.

The unitary transformations are

$$U_\theta = e^{-\mathrm{i}P\theta} \tag{B.5}$$

$$V_\theta = e^{\mathrm{i}Q\theta} \tag{B.6}$$

After these unitary transformation, the reference state is changed as

$$\rho_\theta = U_\theta \rho_0 U_\theta^\dagger \tag{B.7}$$

$$\rho_\theta = V_\theta \rho_0 V_\theta^\dagger \tag{B.8}$$

B.1 Variance of P and Q

B.1.1 Coherent state

The coherent state is defined by $A|z\rangle = z|z\rangle$. With using the number state $|n\rangle$ defined by

$$A^\dagger A|n\rangle = n|n\rangle \tag{B.9}$$

$|n\rangle$ satisfies the following relations.

$$\begin{aligned} A|n\rangle &= \sqrt{n}|n\rangle \\ A^\dagger|n\rangle &= \sqrt{n+1}|n\rangle \end{aligned}$$

$|z\rangle$ is written as

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \tag{B.10}$$

Below we show $A|z\rangle = z|z\rangle$.

$$\begin{aligned} A|z\rangle &= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} A|n\rangle \\ &= \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle \\ &= \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{(n-1)!}}|n-1\rangle \\ &= \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}|k\rangle \\ &= |z\rangle \end{aligned} \tag{B.11}$$

B.1.2 Variance

We write the expectation value of P with respect to the state ρ_0 as $\langle P\rangle_0 = \mathrm{tr}[\rho_0 P]$. Then, the varicane of P, $(\Delta P)^2$ is

$$(\Delta P)^2 = \langle (P - \langle P\rangle_0)^2\rangle_0 = \langle P^2\rangle_0 - \langle P\rangle_0^2 \tag{B.12}$$

$$\begin{aligned} \langle P\rangle_0 &= -\frac{\mathrm{i}}{\sqrt{2c}}\langle A - A^\dagger\rangle_0 = -\frac{\mathrm{i}}{\sqrt{2c}}(z - z^*) \\ \langle P\rangle_0^2 &= -\frac{1}{2}\{z^2 - 2|z|^2 + (z^*)^2\} \end{aligned} \tag{B.13}$$

$$\begin{aligned}\langle P^2\rangle_0 &= -\frac{1}{2c}\langle AA - 2A^\dagger A + 1 + A^\dagger A^\dagger\rangle_{\rho_0} \\ &= -\frac{1}{2c}\{z^2 - 4\kappa^2 - 2|z|^2 + 1 + (z^*)^2\} \qquad \text{(B.14)}\end{aligned}$$

$$(\Delta P)^2 = \langle (P - \langle P\rangle_0)^2\rangle_0 = \frac{1}{2c}(1 + 4\kappa^2) \qquad \text{(B.15)}$$

$(\Delta Q)^2$ can be calculated in the same way and the result is

$$(\Delta Q)^2 = \frac{c}{2}(1 + 4\kappa^2) \qquad \text{(B.16)}$$

We used the results for $\mathrm{tr}\,[\rho_0 A],\ \mathrm{tr}\,[\rho_0 A^\dagger],\ \mathrm{tr}\,[\rho_0 AA],\ \mathrm{tr}\,[\rho_0 A^\dagger A^\dagger],$ and $\mathrm{tr}\,[\rho_0 A^\dagger A^\dagger]$.

$$\begin{aligned}\mathrm{tr}\,[\rho_0 A] &= \frac{1}{2\pi\kappa^2}\frac{1}{\pi}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,\langle\alpha|\,A\,|z'\rangle\,\langle z'|\alpha\rangle\,d^2z'\,d\alpha \\ &= \frac{1}{2\pi\kappa^2}\frac{1}{\pi}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,\langle z'|\alpha\rangle\,\langle\alpha|\,A\,|z'\rangle\,d^2z'\,d\alpha \\ &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,\langle z'|\,A\,|z'\rangle\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,z'\,\langle z'|\,z'\rangle\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int z' e^{-\frac{|z'-z|^2}{2\kappa^2}}\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int (z' + z)e^{-\frac{|z'|^2}{2\kappa^2}}\,d^2z' \\ &= z\end{aligned}$$

$$\mathrm{tr}\,[\rho_0 A^\dagger] = (\mathrm{tr}\,[\rho_0 A])^* = z^*$$

$$\begin{aligned}\mathrm{tr}\,[\rho_0 AA] &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,\langle z'|\,AA\,|z'\rangle\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,(z')^2\,\langle z'|\,z'\rangle\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int (z')^2 e^{-\frac{|z'-z|^2}{2\kappa^2}}\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int (z' + z)^2 e^{-\frac{|z'|^2}{2\kappa^2}}\,d^2z' \\ &= \frac{1}{2\pi\kappa^2}\int z^2 e^{-\frac{|z'|^2}{2\kappa^2}}\,d^2z' \\ &= z^2\end{aligned}$$

$$
\begin{aligned}
\mathrm{tr}\,[\rho_0 A^\dagger A^\dagger] &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\langle z'|\,A^\dagger A^\dagger\,|z'\rangle\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}(z'^{\,*})^2\langle z'|\,z'\rangle\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (z'^{\,*})^2 e^{-\frac{|z'-z|^2}{2\kappa^2}}\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (z'^{\,*}+z^*)^2 e^{-\frac{|z'|^2}{2\kappa^2}}\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (z^*)^2 e^{-\frac{|z'|^2}{2\kappa^2}}\, d^2z' \\
&= (z^*)^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}\,[\rho_0 A^\dagger A] &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\langle z'|\,A^\dagger A\,|z'\rangle\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}} z'^* z'\langle z'|\,z'\rangle\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int z'^* z' e^{-\frac{|z'-z|^2}{2\kappa^2}}\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (z'+z)^*(z'+z) e^{-\frac{|z'|^2}{2\kappa^2}}\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (|z'|^2+|z|^2) e^{-\frac{|z'|^2}{2\kappa^2}}\, d^2z' \\
&= 2\kappa^2+|z|^2
\end{aligned}
$$

B.2 About the method of derivation

The way used for the derivation of RLD and SLD with respect to $\rho_\theta = U_\theta\rho_0 U_\theta^\dagger$ is explained in the following.

1. Derive commutation and anti-commutation relation of $A, A^\dagger$ and ρ_0.
2. Fisrt, calculate $\dfrac{d\rho_\theta}{d\theta} = \dfrac{d}{d\theta}(U_\theta\rho_0 U_\theta^\dagger)$, then calculate RLD L_θ^R by using the result obtained at 1.
3. With using the results obtained at 1, derive RLD from SLD L_θ^S.
4. Confirm the result at 2 and 3 are the same.

B.3 Calculations

B.3.1 $\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$

① Commutation and anti-commutation relation of $A, A^\dagger$ and ρ_0

Since the reference state is Gaussian state, ρ_0 is expressed as

$$\rho_0 = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}} |z'\rangle\langle z'| \, d^2z'$$

By applying A from the left, we have

$$A^\dagger\rho_0 = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}} A^\dagger |z'\rangle\langle z'| \, d^2z' = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}} (z'^* + \frac{\partial}{\partial z'}) |z'\rangle\langle z'| \, d^2z' \tag{B.17}$$

The following result was used.

$$\begin{aligned}
\frac{\partial}{\partial z}|z\rangle\langle z| &= \frac{\partial}{\partial z}\{e^{-|z|^2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^n}{\sqrt{n!}}\frac{(z^*)^k}{\sqrt{k!}}|n\rangle\langle k|\} \\
&= -z^*|z\rangle\langle z| + e^{-|z|^2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{nz^{n-1}}{\sqrt{n!}}\frac{(z^*)^k}{\sqrt{k!}}|n\rangle\langle k| \\
&= -z^*|z\rangle\langle z| + e^{-|z|^2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(m+1)z^m}{\sqrt{(m+1)!}}\frac{(z^*)^k}{\sqrt{k!}}|m+1\rangle\langle k| \\
&= -z^*|z\rangle\langle z| + e^{-|z|^2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\sqrt{(m+1)}z^m}{\sqrt{m!}}\frac{(z^*)^k}{\sqrt{k!}}|m+1\rangle\langle k| \\
&= -z^*|z\rangle\langle z| + e^{-|z|^2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^m}{\sqrt{m!}}\frac{(z^*)^k}{\sqrt{k!}}A^\dagger|m\rangle\langle k| \\
&= -z^*|z\rangle\langle z| + A^\dagger e^{-|z|^2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^m}{\sqrt{m!}}\frac{(z^*)^k}{\sqrt{k!}}|m\rangle\langle k| \\
&= (-z^* + A^\dagger)|z\rangle\langle z|
\end{aligned}$$

$$\therefore A^\dagger|z\rangle\langle z| = (z^* + \frac{\partial}{\partial z})|z\rangle\langle z| \tag{B.18}$$

$\int e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{\partial}{\partial z'}|z'\rangle\langle z'| \, d^2z'$ in (B.18) is calculate as follows.

Let $z' = x' + \mathrm{i}y'$. With using, $\frac{\partial}{\partial z'} = \frac{1}{2}(\frac{\partial}{\partial x'} - \mathrm{i}\frac{\partial}{\partial y'})$,

$$\frac{\partial}{\partial z'}|z'\rangle\langle z'| = \frac{1}{2}(\frac{\partial}{\partial x'} - \mathrm{i}\frac{\partial}{\partial y'})|z'\rangle\langle z'|$$

$$
\begin{aligned}
\int e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{\partial}{\partial z'} |z'\rangle \langle z'| \, d^2 z' &= \int e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{1}{2}\left(\frac{\partial}{\partial x'} - \mathrm{i}\frac{\partial}{\partial y'}\right) |z'\rangle \langle z'| \, dx' dy' \\
&= \int \left[e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{1}{2} |z'\rangle \langle z'| \right]_{x'=-\infty}^{x'=\infty} dy' \\
&\quad - \int \frac{1}{2}\left(\frac{\partial}{\partial x'} e^{-\frac{|z'-z|^2}{2\kappa^2}}\right) |z'\rangle \langle z'| \, dx' dy' \\
&\quad - \mathrm{i} \int \left[e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{1}{2} |z'\rangle \langle z'| \right]_{y'=-\infty}^{y'=\infty} dx' \\
&\quad + \mathrm{i} \int \frac{1}{2}\left(\frac{\partial}{\partial y'} e^{-\frac{|z'-z|^2}{2\kappa^2}}\right) |z'\rangle \langle z'| \, dx' dy' \\
&= - \int \frac{1}{2}\left\{\left(\frac{\partial}{\partial x'} - \mathrm{i}\frac{\partial}{\partial y'}\right) e^{-\frac{|z'-z|^2}{2\kappa^2}}\right\} |z'\rangle \langle z'| \, dx' dy' \\
&= - \int \left(\frac{\partial}{\partial z'} e^{-\frac{|z'-z|^2}{2\kappa^2}}\right) |z'\rangle \langle z'| \, d^2 z' \\
&= - \int \left(-\frac{z'^* - z^*}{2\kappa^2}\right) e^{-\frac{|z'-z|^2}{2\kappa^2}} |z'\rangle \langle z'| \, d^2 z' \\
&= \frac{1}{2\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} |z'\rangle \langle z'| (A^\dagger - z^*) \, d^2 z' \\
&= \frac{1}{2\kappa^2} 2\pi\kappa^2 \rho_0 (A^\dagger - z^*)
\end{aligned}
\tag{B.19}
$$

Since the annihilation operator A acts on $|z\rangle$,

$$
\begin{aligned}
A^\dagger \rho_0 &= \frac{1}{2\pi\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} A \, |z'\rangle \langle z'| \, d^2 z' \\
&= \frac{1}{2\pi\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} \left(z'^* + \frac{\partial}{\partial z'}\right) |z'\rangle \langle z'| \, d^2 z'
\end{aligned}
$$

With using

$$
z^{*\prime} |z'\rangle \langle z'| = |z'\rangle \langle z'| z^{*\prime} = |z'\rangle \langle z'| A^\dagger,
$$

we have

$$
\frac{1}{2\pi\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} z'^* |z'\rangle \langle z'| \, d^2 z' = \frac{1}{2\pi\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} |z'\rangle \langle z'| A^\dagger d^2 z' = \rho_0 A^\dagger.
$$

Therefore,

$$
A^\dagger \rho_0 = \rho_0 A^\dagger + \frac{1}{2\pi\kappa^2} \int e^{-\frac{|z'-z|^2}{2\kappa^2}} \frac{\partial}{\partial z'} |z'\rangle \langle z'| \, d^2 z'
$$

From (B.18),

$$
\begin{aligned}
A^\dagger \rho_0 &= \rho_0 A^\dagger + \frac{1}{2\kappa^2} \rho_0 (A^\dagger - z^*) \\
&= \frac{1}{2\kappa^2} \rho_0 \{(2\kappa^2 + 1) A^\dagger - z^*\}
\end{aligned}
\tag{B.20}
$$

By taking the Hermite conjugate of $(B.20)$,

$$\rho_0 A = \frac{1}{2\kappa^2}\{(2\kappa^2+1)A - z\}\rho_0 \tag{B.21}$$

From $(B.21)$,

$$\begin{aligned}
\rho_0(A + \frac{1}{2\kappa^2}z) &= (1+\frac{1}{2\kappa^2})A\,\rho_0 \\
A\rho_0 &= \frac{1}{1+\frac{1}{2\kappa^2}}\,\rho_0\,(A + \frac{1}{2\kappa^2}z) \\
A\rho_0 &= \frac{2\kappa^2}{1+2\kappa^2}\,\rho_0\,(A + \frac{1}{2\kappa^2}z)
\end{aligned}$$

$$\therefore A\rho_0 = \frac{1}{1+2\kappa^2}\,\rho_0\,(2\kappa^2 A + z) \tag{B.22}$$

From $(B.22)$,

$$\{A, \rho_0\} = \frac{1}{1+2\kappa^2}\,\rho_0\,(2\kappa^2 A + z) + \rho_0 A$$

$$\{A, \rho_0\} = \frac{1}{1+2\kappa^2}\,\rho_0\,\{(1+4\kappa^2)A + z\} \tag{B.23}$$

② Calculation of $\dfrac{d\rho_\theta}{d\theta} = \dfrac{d}{d\theta}(U_\theta \rho_0 U_\theta^\dagger)$

The right hand side, $\dfrac{d}{d\theta}(U_\theta \rho_0 U_\theta^\dagger)$ is

$$\begin{aligned}
\frac{d\rho_\theta}{d\theta} &= \frac{dU_\theta}{d\theta}\rho_0 U^\dagger + U_\theta \rho_0 \frac{dU_\theta^\dagger}{d\theta} \\
&= -\mathrm{i}PU_\theta\rho_0 U_\theta^\dagger + \mathrm{i}U_\theta\rho_0 U_\theta^\dagger P \\
&= -\mathrm{i}U_\theta U_\theta^\dagger P U_\theta \rho_0 U_\theta^\dagger + \mathrm{i}U_\theta\rho_0 U_\theta^\dagger P U_\theta U_\theta^\dagger
\end{aligned}$$

From $U_\theta P U_\theta^\dagger = P$,

$$\frac{d\rho_\theta}{d\theta} = -\mathrm{i}U_\theta P \rho_0 U_\theta^\dagger + \mathrm{i}U_\theta \rho_0 P U_\theta^\dagger$$

$$\frac{d\rho_\theta}{d\theta} = -\mathrm{i}U_\theta[P, \rho_0]U_\theta^\dagger \tag{B.24}$$

$A = \dfrac{1}{\sqrt{c}}(Q + \mathrm{i}cP),\ A^\dagger = \dfrac{1}{\sqrt{c}}(Q - \mathrm{i}cP)$ give $P = \dfrac{1}{\mathrm{i}\sqrt{2c}}(A - A^\dagger)$.

$$-\mathrm{i}[P, \rho_0] = -\mathrm{i}\,\frac{1}{\mathrm{i}\sqrt{2c}}[A - A^\dagger, \rho_0] = -\frac{1}{\sqrt{2c}}[A - A^\dagger, \rho_0]$$

$$\therefore \frac{d\rho_\theta}{d\theta} = -\mathrm{i}U_\theta[P, \rho_0]U_\theta^\dagger = -\frac{1}{\sqrt{2c}}U_\theta[A - A^\dagger, \rho_0]U_\theta^\dagger \tag{B.25}$$

$$
\begin{aligned}
[A - A^\dagger, \rho_0] &= A\rho_0 - A^\dagger\rho_0 - \rho_0 A + \rho_0 A^\dagger \\
&= \rho_0[\frac{2\kappa^2 A + z}{1+2\kappa^2} - \{(1+\frac{1}{2\kappa^2})A^\dagger - \frac{z^*}{2\kappa^2}\} - A + A^\dagger] \\
&= \rho_0\{(\frac{2\kappa^2}{1+2\kappa^2} - 1)A - \frac{1}{2\kappa^2}A^\dagger + \frac{z}{1+2\kappa^2} + \frac{z^*}{2\kappa^2}\} \\
&= \rho_0(-\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger + \frac{z}{1+2\kappa^2} + \frac{z^*}{2\kappa^2})
\end{aligned}
$$

From (B.25),

$$
\begin{aligned}
\frac{d\rho_\theta}{d\theta} &= U_\theta\rho_0(-\frac{1}{\sqrt{2c}})(-\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger + \frac{z}{1+2\kappa^2} + \frac{z^*}{2\kappa^2})U_\theta^\dagger \\
&= U_\theta\rho_0 U_\theta^\dagger U_\theta \frac{1}{\sqrt{2c}}(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger - \frac{z}{1+2\kappa^2} - \frac{z^*}{2\kappa^2})U_\theta^\dagger \\
&= \rho_\theta U_\theta \frac{1}{\sqrt{2c}}(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger - \frac{z}{1+2\kappa^2} - \frac{z^*}{2\kappa^2})U_\theta^\dagger
\end{aligned}
$$

Therefore, if L_θ^R is defined by the equation, $\frac{d\rho_\theta}{d\theta} = \rho_\theta L_\theta^R$, L_θ^R is

$$
L_\theta^R = U_\theta \frac{1}{\sqrt{2c}}(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger - \frac{z}{1+2\kappa^2} - \frac{z^*}{2\kappa^2})U_\theta^\dagger \tag{B.26}
$$

$$
L_\theta^R = \frac{1}{\sqrt{2c}}\{\frac{1}{1+2\kappa^2}(A - \frac{1}{\sqrt{2c}}\theta) + \frac{1}{2\kappa^2}(A^\dagger - \frac{1}{\sqrt{2c}}\theta) - \frac{z}{1+2\kappa^2} - \frac{z^*}{2\kappa^2}\} \tag{B.27}
$$

$$
\because\ U_\theta A U_\theta^\dagger = U_\theta \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP)U_\theta^\dagger = \frac{1}{\sqrt{2c}}(Q - \theta + \mathrm{i}cP) = A - \frac{1}{\sqrt{2c}}\theta
$$

③ Derive RLD from SLD L_θ^S for $\rho_\theta = U_\theta\rho_0 U_\theta^\dagger$

$$
\{A + A^\dagger, \rho_0\} = \rho_0\{(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger) + \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*\} \tag{B.28}
$$

$A = \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP)$ and $A^\dagger = \frac{1}{\sqrt{2c}}(Q - \mathrm{i}cP)$ give $Q = \sqrt{\frac{c}{2}}(A + A^\dagger)$

$$
\{Q, \rho_0\} = \rho_0\{\sqrt{\frac{c}{2}}(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger) + \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*\} \tag{B.29}
$$

From $L_\theta^S = \frac{2}{c(1+4\kappa^2)}(Q - \langle Q\rangle_\rho)$,

$$
\begin{aligned}
\frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S\rho_\theta) &= \frac{1}{2}\{L_\theta^S,\ \rho_\theta\} \\
&= \frac{1}{c(1+4\kappa^2)}(\{Q,\ \rho_\theta\} - 2\rho_\theta\langle Q\rangle_\rho)
\end{aligned} \tag{B.30}
$$

$$
\begin{aligned}
\{Q,\, \rho_\theta\} &= Q\rho_\theta + \rho_\theta Q \\
&= QU_\theta\rho_0 U_\theta^\dagger + U_\theta\rho_0 U_\theta^\dagger Q \\
&= U_\theta U_\theta^\dagger Q U_\theta \rho_0 U_\theta^\dagger + U_\theta\rho_0 U_\theta^\dagger Q U_\theta U_\theta^\dagger
\end{aligned}
$$

With using $U_\theta^\dagger Q U_\theta = Q + \theta$,

$$
\begin{aligned}
\{Q,\, \rho_\theta\} &= U_\theta(Q+\theta)\rho_0 U_\theta^\dagger + U_\theta\rho_0(Q+\theta)U_\theta^\dagger \\
&= U_\theta(Q\rho_0 + \rho_0 Q + 2\theta)U_\theta^\dagger \\
&= U_\theta\{Q,\, \rho_0\}U_\theta^\dagger + 2\theta U_\theta\rho_0 U_\theta^\dagger
\end{aligned}
$$

$$
\begin{aligned}
\{Q,\, \rho_\theta\} - 2\rho_\theta\langle Q\rangle_\rho &= U_\theta\{Q,\, \rho_0\}U_\theta^\dagger + 2\theta U_\theta\rho_0 U_\theta^\dagger - 2\rho_\theta\langle Q\rangle_\rho \\
&= U_\theta\{Q,\, \rho_0\}U_\theta^\dagger + 2\theta\rho_\theta - 2\rho_\theta(\langle Q\rangle_0 + \theta) \\
&= U_\theta(\{Q,\, \rho_0\} - 2\rho_0\langle Q\rangle_0)U_\theta^\dagger
\end{aligned}
\tag{B.31}
$$

$$
\frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S\rho_\theta) = \frac{1}{c(1+4\kappa^2)}(\{Q, \rho_\theta\} - 2\rho_\theta\langle Q\rangle_\rho) = U_\theta\frac{1}{c(1+4\kappa^2)}(\{Q, \rho_0\} - 2\rho_0\langle Q\rangle_0)U_\theta^\dagger \tag{B.32}
$$

$$
\begin{aligned}
\langle Q\rangle_0 &= \mathrm{tr}\,[\rho_0 Q] = \sqrt{\frac{c}{2}}\mathrm{tr}\,[\rho_0(A + A^\dagger)] \\
\langle A\rangle_0 &= \mathrm{tr}\,[\rho_0 A] \\
&= \frac{1}{2\pi\kappa^2}\frac{1}{\pi}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,\langle\alpha|A|\,z'\rangle\,\langle z'|\,\alpha\rangle\, d^2z'd\alpha \\
&= \frac{1}{2\pi\kappa^2}\frac{1}{\pi}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,z'\,\langle\alpha|\,z'\rangle\,\langle z'|\,\alpha\rangle\, d^2z'd\alpha \\
&= \frac{1}{2\pi\kappa^2}\frac{1}{\pi}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,z'\,\langle z'|\,\alpha\rangle\,\langle\alpha|\,z'\rangle\, d^2z'd\alpha \\
&= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'-z|^2}{2\kappa^2}}\,z'\,\langle z'|\,z'\rangle\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int z' e^{-\frac{|z'-z|^2}{2\kappa^2}}\, d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int (z'+z) e^{-\frac{|z'|^2}{2\kappa^2}}\, d^2z' \\
&= z \\
\langle A^\dagger\rangle_0 &= \mathrm{tr}\,[\rho_0 A^\dagger] = \mathrm{tr}\,[\rho_0 A]^* = z^*
\end{aligned}
$$

$$
\langle Q\rangle_0 = \sqrt{\frac{c}{2}}(z + z^*) \tag{B.33}
$$

$$\frac{1}{2}(\rho_\theta L^S_\theta + L^S_\theta\rho_\theta) = U_\theta\frac{1}{c(1+4\kappa^2)}\rho_0[\sqrt{\frac{c}{2}}\{(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger) + \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*\} - 2\rho_0\sqrt{\frac{c}{2}}(z+z^*)]U$$

$$= \frac{1}{c}\sqrt{\frac{c}{2}}U_\theta\rho_0[\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger + \frac{1}{1+4\kappa^2}\{\frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^* - 2(z+z^*)\}]U^\dagger_\theta$$

$$= \frac{1}{\sqrt{2c}}U_\theta\rho_0\{\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger - \frac{1+4\kappa^2}{1+4\kappa^2}(\frac{1}{1+2\kappa^2}z + \frac{1}{2\kappa^2}z^*)\}U^\dagger_\theta$$

$$\frac{1}{2}(\rho_\theta L^S_\theta + L^S_\theta\rho_\theta) = \frac{1}{\sqrt{2c}}U_\theta\rho_0\{\frac{1}{1+2\kappa^2}A + \frac{1}{2\kappa^2}A^\dagger - \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*)\}U^\dagger_\theta \tag{B.34}$$

④ for for U_θ Confirm that the results of ② and ③ are the same.

With (B.26) and L^R_θ, the right hand side of (B.32) is expressed as

$$\frac{1}{2}(\rho_\theta L^S_\theta + L^S_\theta\rho_\theta) = \rho_\theta L^R_\theta \tag{B.35}$$

Therefore,

$$\rho_\theta L^R_\theta = \frac{d\rho_\theta}{d\theta} = \frac{1}{2}(\rho_\theta L^S_\theta + L^S_\theta\rho_\theta) \tag{B.36}$$

B.3.2 $\rho_\theta = V_\theta\rho_0 V^\dagger_\theta$

①. No change. The results are applicable to this case.

② $\frac{d\rho_\theta}{d\theta} = \frac{d}{d\theta}(V_\theta\rho_0 V^\dagger_\theta)$

The left hand side, $\frac{d\rho_\theta}{d\theta}$ is

$$\begin{aligned}\frac{d\rho_\theta}{d\theta} &= \frac{dV_\theta}{d\theta}\rho_0 V^\dagger + V_\theta\rho_0\frac{dV^\dagger_\theta}{d\theta}\\ &= \mathrm{i}QV_\theta\rho_0V^\dagger_\theta + \mathrm{i}V_\theta\rho_0V^\dagger_\theta(-\mathrm{i}Q)\\ &= \mathrm{i}V_\theta[Q,\rho_0]V^\dagger_\theta\end{aligned}$$

$$\frac{d\rho_\theta}{d\theta} = -\mathrm{i}U_\theta[P,\rho_0]U^\dagger_\theta$$

$A = \frac{1}{\sqrt{c}}(Q + \mathrm{i}cP)$ and $A^\dagger = \frac{1}{\sqrt{c}}(Q - \mathrm{i}cP)$ give $Q = \sqrt{\frac{c}{2}}(A + A^\dagger)$

$$\frac{d\rho_\theta}{d\theta} = \mathrm{i}\sqrt{\frac{c}{2}}V_\theta[A + A^\dagger,\rho_0]V^\dagger_\theta$$

$$
\begin{aligned}
[A+A^\dagger,\rho_0] &= A\rho_0-\rho_0A+A^\dagger\rho_0-\rho_0A^\dagger\\
&= \rho_0[\frac{2\kappa^2A+z}{1+2\kappa^2}+\{(1+\frac{1}{2\kappa^2})A^\dagger-\frac{z^*}{2\kappa^2}\}-A-A^\dagger]\\
&= \rho_0\{(\frac{2\kappa^2}{1+2\kappa^2}-1)A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2}\}\\
&= \rho_0(-\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2})
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \frac{d\rho_\theta}{d\theta} &= V_\theta\rho_0(\mathrm{i}\sqrt{\frac{c}{2}})\rho_0(-\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2})V_\theta^\dagger\\
&= V_\theta\rho_0V_\theta^\dagger V_\theta(\mathrm{i}\sqrt{\frac{c}{2}})\rho_0(-\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2})V_\theta^\dagger\\
&= \rho_\theta V_\theta(\mathrm{i}\sqrt{\frac{c}{2}})\rho_0(-\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2})V_\theta^\dagger
\end{aligned}
$$

Therefore, L_θ^R is

$$
L_\theta^R = V_\theta(\mathrm{i}\sqrt{\frac{c}{2}})\rho_0(-\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger+\frac{z}{1+2\kappa^2}-\frac{z^*}{2\kappa^2})V_\theta^\dagger \tag{B.37}
$$

③ Derive RLD from SLD L_θ^S for $\rho_\theta = V_\theta\rho_0V_\theta^\dagger$ The RLD and SLD for $\rho_\theta = V_\theta\rho_0V_\theta^\dagger$ is also calculated in the same way.

$$
\{A+A^\dagger,\rho_0\} = \rho_0\{(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger)+\frac{1}{1+2\kappa^2}z-\frac{1}{2\kappa^2}z^*\} \tag{B.38}
$$

From $A=\frac{1}{\sqrt{2c}}(Q+\mathrm{i}cP)$, $A^\dagger=\frac{1}{\sqrt{2c}}(Q-\mathrm{i}cP)$, and $Q=\sqrt{\frac{c}{2}}(A+A^\dagger)$,

$$
\{Q,\rho_0\} = \rho_0\{\sqrt{\frac{c}{2}}(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A+\frac{1}{2\kappa^2}A^\dagger)+\frac{1}{1+2\kappa^2}z-\frac{1}{2\kappa^2}z^*\} \tag{B.39}
$$

We can calculate in the same way as above with using $L_\theta^S=\frac{2c}{1+4\kappa^2}(P-\langle P\rangle_0)$.

$$
\begin{aligned}
\frac{1}{2}(\rho_\theta L_\theta^S+L_\theta^S\rho_\theta) &= \frac{1}{2}\{L_\theta^S,\ \rho_\theta\}\\
&= \frac{c}{1+4\kappa^2}(\{P,\ \rho_\theta\}-2\rho_\theta\langle P\rangle_0)\\
&= U_\theta\frac{c}{1+4\kappa^2}(\{P,\ \rho_0\}-2\rho_0\langle P\rangle_0)U_\theta^\dagger
\end{aligned}
$$

$\{P,\, \rho_0\}$ can be calculated as

$$
\begin{aligned}
\{P,\, \rho_0\} &= -\frac{\mathrm{i}}{\sqrt{2c}}\{A - A^\dagger,\, \rho_0\} \\
&= -\frac{\mathrm{i}}{\sqrt{2c}}(\{A\, \rho_0\} - \{A^\dagger,\, \rho_0\})
\end{aligned}
$$

From (B.29),

$$
-\frac{\mathrm{i}}{\sqrt{2c}}\{A - A^\dagger,\, \rho_0\} = -\frac{\mathrm{i}}{\sqrt{2c}}(1 + 4\kappa^2)\rho_0(\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger - \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*)
$$

$$
\frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S \rho_\theta) = -\frac{\mathrm{i}}{\sqrt{2c}}(1 + 4\kappa^2)\rho_0(\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger - \frac{1}{1+2\kappa^2}z - \frac{1}{2\kappa^2}z^*)
$$

$\langle P\rangle_0$ is calculated as

$$
\begin{aligned}
\langle P\rangle_0 &= tr[\rho_0 P] = -\frac{\mathrm{i}}{\sqrt{2c}}tr[\rho_0(A - A^\dagger)] \\
\langle A\rangle_0 &= tr[\rho_0 A] = z \\
\langle A^\dagger\rangle_0 &= z^*
\end{aligned}
$$

$$
\langle P\rangle_0 = -\frac{\mathrm{i}}{\sqrt{2c}}(z - z^*)
$$

$$
\begin{aligned}
\frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S \rho_\theta) &= V_\theta \frac{c}{1+4\kappa^2}\rho_0[(-\frac{\mathrm{i}}{\sqrt{2c}})\{(1+4\kappa^2)(\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger) + \frac{1}{1+2\kappa^2}z + \frac{1}{2\kappa^2}z^*\} - 2\rho_0(z - z^*)]V_\theta^\dagger \\
&= -\frac{\mathrm{i}}{\sqrt{2c}}V_\theta \rho_0\{\frac{1}{1+2\kappa^2}A - \frac{1}{2\kappa^2}A^\dagger - \frac{1}{1+2\kappa^2}z + \frac{1}{2\kappa^2}z^*\}V_\theta^\dagger
\end{aligned}
$$

④ To confirm that the results of ② and ③ are the same.

From (B.37) and with using L_θ^R, the equation above is expressed as

$$
\frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S \rho_\theta) = \rho_\theta L_\theta^R \tag{B.40}
$$

Therefore,

$$
\rho_\theta L_\theta^R = \frac{d\rho_\theta}{d\theta} = \frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S \rho_\theta) \tag{B.41}
$$

B.4 SLD Fisher information

By the definition, $g^S(\theta) = \|L_\theta\|^2 = (L_\theta, L_\theta)_\theta = \mathrm{Re}(\mathrm{tr}[\rho_0 L_0 L_0^\dagger])$.

For the Fisher information of $\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$ is

$$\begin{aligned}
L_0^S &= \frac{2}{c(1+4\kappa^2)}(Q - \langle Q \rangle_0) \\
g_\theta^S &= \mathrm{tr}[\rho_0 L_0^S L_0^{S\dagger}] \\
&= \frac{4}{c^2(1+4\kappa^2)^2}\mathrm{tr}[\rho(Q - \langle Q \rangle_0)(Q - \langle Q \rangle_0)] \\
&= \frac{4}{c^2(1+4\kappa^2)^2}(\Delta Q)^2 \\
&= \frac{4}{c^2(1+4\kappa^2)^2}\frac{2}{c}(1+4\kappa^2) \\
&= \frac{2}{c^2(1+4\kappa^2)} \\
\therefore g_\theta^S &= \mathrm{Re}\,(\mathrm{tr}[\rho_0 L_0^S L_0^{S\dagger}]) = \frac{2}{c(1+4\kappa^2)}
\end{aligned} \tag{B.42}$$

For the fisher information of $\rho_\theta = V_\theta \rho_0 V_\theta^\dagger$ is

$$\begin{aligned}
L_0^S &= \frac{2c}{1+4\kappa^2}(P - \langle P \rangle_0) \\
g_\theta^S &= \mathrm{tr}[\rho_0 L_0^S L_0^{S\dagger}] \\
&= \frac{4c^2}{(1+4\kappa^2)^2}\mathrm{tr}[\rho(P - \langle P \rangle_0)(P - \langle P \rangle_0)] \\
&= \frac{2c}{1+4\kappa^2} \\
\therefore g_\theta^S &= \mathrm{Re}\,(\mathrm{tr}[\rho_0 L_0^S L_0^{S\dagger}]) = \frac{2c}{1+4\kappa^2}
\end{aligned} \tag{B.43}$$

Appendix C

Two-parameter estimation : Gaussian model

Q and P are the position and the momentum operators, respectively.

$$[Q, P] = QP - PQ = \mathrm{i} \tag{C.1}$$

The operator called annihilation operator, A is defined as

$$A = \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP) \tag{C.2}$$

The Hermite conjugate of A, $A^\dagger$ and A have a commutation relation,

$$[A, A^\dagger] = 1 \tag{C.3}$$

$$\begin{aligned}
\because\ [A, A^\dagger] &= \frac{1}{2c}[Q + \mathrm{i}cP, Q - \mathrm{i}cP] \\
&= \frac{1}{2c}([Q, -\mathrm{i}cP] + [\mathrm{i}cP, Q]) \\
&= -\frac{2\mathrm{i}c}{2c}[Q, P] \\
&= -\mathrm{i} \times \mathrm{i} = 1
\end{aligned}$$

We choose the Gaussian state as the reference state.

$$\rho_0 = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'|^2}{2\kappa^2}} |z'\rangle\langle z'|\, d^2z' \tag{C.4}$$

where $|z\rangle$ is called the coherent state which is defined by $A|z\rangle = z|z\rangle$

$$A = \frac{1}{\sqrt{2c}}(Q + \mathrm{i}cP) \tag{C.5}$$

Unitary Transformations are

$$U_{\theta^1} = \mathrm{e}^{-\mathrm{i}P\theta^1}$$
$$V_{\theta^2} = \mathrm{e}^{\mathrm{i}Q\theta^2}$$

ρ_{θ^1} and ρ_{θ^2} are the states after the transformation by U_{θ^1} and V_{θ^2}, respectively.

$$\rho_{\theta^1} = U_{\theta^1}\rho_0 U_{\theta^1}^\dagger$$
$$\rho_{\theta^2} = V_{\theta^2}\rho_0 V_{\theta^2}^\dagger$$

The state after transformed by U_{θ^1} and V_{θ^2}, ρ_θ is

$$\rho_\theta = U_{\theta^1} V_{\theta^2}\rho_0 U_{\theta^1}^\dagger V_{\theta^2}^\dagger \tag{C.6}$$

$$\rho_\theta = U_{\theta^1} V_{\theta^2}\rho_0 (U_{\theta^1} V_{\theta^2})^\dagger = U_{\theta^1} V_{\theta^2}\rho_0 V_{\theta^2}^\dagger U_{\theta^1}^\dagger = e^{-\mathrm{i}\theta^1\theta^2} V_{\theta^2} U_{\theta^1}\rho_0 e^{\mathrm{i}\theta^1\theta^2} U_{\theta^1} V_{\theta^2} = V_{\theta^2} U_{\theta^1}\rho_0 U_{\theta^1}^\dagger V_{\theta^2}^\dagger \tag{C.7}$$

$U_{\theta^1} V_{\theta^2} = e^{-\mathrm{i}\theta^1\theta^2} V_{\theta^2} U_{\theta^1}$ is derived as follows.

$$\begin{aligned} U_{\theta^1}\psi(x) = e^{-\mathrm{i}P\theta^1}\psi(x) &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}P\theta^1)^n}{n!}\psi(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\{\frac{d}{dx}(-\theta^1)\}^n \psi(x) \\ &= \psi(x-\theta^1) \end{aligned}$$

$$V_{\theta^2}\psi(x) = e^{\mathrm{i}\theta^2 x}\psi(x) \tag{C.8}$$

$$\therefore\ U_{\theta^1} V_{\theta^2}\psi(x) = U_{\theta^1} e^{\mathrm{i}\theta^2 x}\psi(x) = e^{\mathrm{i}\theta^2(x-\theta^1)}\psi(x-\theta^1) = e^{-\mathrm{i}\theta^1\theta^2} e^{\mathrm{i}\theta^2 x}\psi(x-\theta^1) = e^{-\mathrm{i}\theta^1\theta^2} V_{\theta^2} U_{\theta^1}\psi(x) \tag{C.9}$$

C.1 Two-parameter SLD

Below, we show here we can calculate L^{1S} and L^{2S} only with U_{θ^1} and V_{θ^2}, respectively.
From (C.7)

$$\rho_\theta = V_{\theta^2} U_{\theta^1}\rho_0 U_{\theta^1}^\dagger V_{\theta^2}^\dagger \tag{C.10}$$

From $\rho_{\theta^1}^1 = U_{\theta^1}\rho_0 U_{\theta^1}^\dagger$,

$$\frac{\partial\rho_\theta}{\partial\theta^1} = V_{\theta^2}(\frac{\partial}{\partial\theta^1} U_{\theta^1}\rho_0 U_{\theta^1}^\dagger) V_{\theta^2}^\dagger = V_{\theta^2}\frac{\partial\rho_{\theta^1}^1}{\partial\theta^1} V_{\theta^2}^\dagger \tag{C.11}$$

With $\dfrac{\partial\rho^1_{\theta^1}}{\partial\theta^1} = \dfrac{1}{2}\{L^S_{\theta^1}, \rho^1_{\theta^1}\}$,

$$\begin{aligned}
\frac{\partial\rho_\theta}{\partial\theta^1} &= V_{\theta^2}\frac{1}{2}\{L^S_{\theta^1}, \rho^1_{\theta^1}\}V^\dagger_{\theta^2} \\
&= \frac{1}{2}(V_{\theta^2}L^S_{\theta^1}\rho^1_{\theta^1}V^\dagger_{\theta^2} + V_{\theta^2}\rho^1_{\theta^1}L^S_{\theta^1}V^\dagger_{\theta^2}) \\
&= \frac{1}{2}(V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}V_{\theta^2}\rho^1_{\theta^1}V^\dagger_{\theta^2} + V_{\theta^2}\rho^1_{\theta^1}V^\dagger_{\theta^2}V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}) \\
&= \frac{1}{2}(V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}\rho_\theta + \rho_\theta V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}) \\
&= \frac{1}{2}\{V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}, \rho_\theta\}
\end{aligned}$$

Define $L^{1S}_{\theta^1}$ by $L^{1S}_{\theta^1} = V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}$

$$\therefore \frac{\partial\rho_\theta}{\partial\theta^1} = V_{\theta^2}\frac{1}{2}\{L^S_{\theta^1}, \rho^1_{\theta^1}\}V^\dagger_{\theta^2} = \frac{1}{2}\{V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2}, \rho_\theta\} = \frac{1}{2}\{L^{1S}_{\theta^1}, \rho_\theta\} \tag{C.12}$$

Let us define $L^S_{\theta,1}$ as

$$\frac{\partial\rho_\theta}{\partial\theta^1} = V_{\theta^2}\frac{1}{2}\{L^S_{\theta^1}, \rho^1_{\theta^1}\}V^\dagger_{\theta^2} = \frac{1}{2}\{L^S_{\theta,1}, \rho_\theta\} \tag{C.13}$$

$$L^S_{\theta,1} = V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2} = L^{1S}_{\theta^1} \tag{C.14}$$

In the mean time, we have

$$L^S_{\theta^1} = \frac{2}{c(1+4\kappa^2)}(Q - \langle Q\rangle_{\theta^1}).$$

We also have $[V_{\theta^2}, Q] = |e^{iQ\theta^2}, Q] = 0$. Therefore, $[V_{\theta^2}, L^S_{\theta^1}] = 0$

$$\therefore L^{1S}_{\theta^1} = V_{\theta^2}L^S_{\theta^1}V^\dagger_{\theta^2} = L^S_{\theta^1}V_{\theta^2}V^\dagger_{\theta^2} = L^S_{\theta^1}$$

$$L^{1S}_{\theta^1} = L^S_{\theta^1} \tag{C.15}$$

In the same way,

$$L^{2S}_{\theta^2} = L^S_{\theta^2} \tag{C.16}$$

Therefore, we can calculate L^{1S} and L^{2S} only with U_{θ^1} and V_{θ^2}, respectively.

C.2 Two-parameter RLD

Below we show the same for RLD as we derived in the previous section.

From $(C.7)$

$$\rho_\theta = V_{\theta^2}U_{\theta^1}\rho_0 U^\dagger_{\theta^1}V^\dagger_{\theta^2} \tag{C.17}$$

Let $\rho^1_{\theta^1} = U_{\theta^1}\rho_0 U^\dagger_{\theta^1}$.

$$\frac{\partial \rho_\theta}{\partial \theta^1} = V_{\theta^2}(\frac{\partial}{\partial \theta^1} U_{\theta^1}\rho_0 U^\dagger_{\theta^1})V^\dagger_{\theta^2} = V_{\theta^2}\frac{\partial \rho^1_{\theta^1}}{\partial \theta^1}V^\dagger_{\theta^2} \tag{C.18}$$

By the definition of RLD,

$$\frac{\partial \rho^1_{\theta^1}}{\partial \theta^1} = \rho^1_{\theta^1} L^R_{\theta^1}. \tag{C.19}$$

$\frac{\partial \rho_\theta}{\partial \theta^1} = V_{\theta^2}\frac{\partial \rho^1_{\theta^1}}{\partial \theta^1}V^\dagger_{\theta^2}$ is calculated as

$$\frac{\partial \rho_\theta}{\partial \theta^1} = V_{\theta^2}\rho^1_{\theta^1}L^R_{\theta^1}V^\dagger_{\theta^2} = V_{\theta^2}\rho^1_{\theta^1}V^\dagger_{\theta^2}V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2} = \rho_\theta V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2}$$

Define $L^{1R}_{\theta^1}$ by $L^{1R}_{\theta^1} = V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2}$

$$\frac{\partial \rho_\theta}{\partial \theta^1} = \rho_\theta V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2} = \rho_\theta L^R_{\theta^1,1}$$

Furthermore, we define $L^R_{\theta,1}$ by $\frac{\partial \rho_\theta}{\partial \theta^1} = \rho_\theta V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2} = \rho_\theta L^R_{\theta^1,1}$. Then, we obtain

$$L^R_{\theta,1} = V_{\theta^2}L^R_{\theta^1}V^\dagger_{\theta^2} = L^R_{\theta^1,1}. \tag{C.20}$$

The same hold for RLD. That is, we can calculate L^R_1 and L^R_2 only with U_{θ^1} and V_{θ^2}, respectively.

C.3 G^R for two-parameter

By the definition, G^R_{11} is

$$G^R_{11} = \mathrm{tr}\,[\rho_0 L^R_{0,1}L^{R\,\dagger}_{0,1}]$$

Let $a = \frac{1}{1+2\kappa^2},\ b = \frac{1}{2\kappa^2}$.

$$L^{1\,R}_0 = \frac{1}{\sqrt{2c}}(aA + bA^\dagger - az - bz^*) \tag{C.21}$$

$$L^{1\,R\,\dagger}_0 = \frac{1}{\sqrt{2c}}(aA^\dagger + bA - az^* - bz) \tag{C.22}$$

$$\begin{aligned}2cL^{1\,R}_0 L^{1\,R\,\dagger}_0 &= a^2AA^\dagger + abA^2 - a^2z^*A - abzA\\ &+ ba(A^\dagger)^2 + b^2A^\dagger A - abz^*A^\dagger - b^2zA^\dagger\\ &- a^2zA^\dagger - abzA - a^2z^*z + abz^2\\ &- abz^*A^\dagger - b^2z^*A - a^2(z^*)^2 + bz^*z\end{aligned}$$

With using the calculation results of $\mathrm{tr}\,[\rho_0 A]$, $\mathrm{tr}\,[\rho_0 A^\dagger]$, $\mathrm{tr}\,[\rho_0 AA]$, $\mathrm{tr}\,[\rho_0 A^\dagger A^\dagger]$, and $\mathrm{tr}\,[\rho_0 A^\dagger A^\dagger]$ in Appendix B.

$$
\begin{aligned}
2cG^R_{11} &= 2c\,\mathrm{tr}\,[\rho_0 L^R_{0,1} L^{R\,\dagger}_{0,1}] \\
&= a^2(z^* z + \frac{1}{b} + 1) + abz^2 - a^2 z^* z - abz^2 \\
&+ ba(z^*)^2 + b^2(z^* z + \frac{1}{b}) - ab(z^*)^2 - b^2 zz^* \\
&- a^2 zz^* - abz^2 - a^2 z^* z + abz^2 \\
&- ab(z^*)^2 - b^2 z^* z - a^2(z^*)^2 + bz^* z \\
&= a^2(\frac{1}{b} + 1) + b \\
&= \frac{1}{(1+2\kappa^2)^2}(1+2\kappa^2) + \frac{1}{2\kappa^2} \\
&= \frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2}
\end{aligned}
$$

$$
\therefore G^R_{11} = \frac{1}{2c}(\frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2}) = \frac{1}{2\kappa^2(1+2\kappa^2)}\frac{1}{2}\frac{(1+4\kappa^2)}{c} \tag{C.23}
$$

G^R_{22} is

$$
G^R_{22} = \mathrm{tr}\,[\rho_0 L^R_{0,2} L^{R\,\dagger}_{0,2}]
$$

$$
L^{2\,R}_0 = \mathrm{i}\sqrt{\frac{c}{2}}(-aA + bA^\dagger + az - bz^*) \tag{C.24}
$$

$$
L^{2\,R\,\dagger}_0 = -\mathrm{i}\sqrt{\frac{c}{2}}(-aA^\dagger + bA + az^* - bz) \tag{C.25}
$$

$$
\begin{aligned}
\frac{2}{c}L^{2\,R}_0 L^{2\,R\,\dagger}_0 &= a^2 AA^\dagger - abA^2 - a^2 z^* A + abzA \\
&- ba(A^\dagger)^2 + b^2 A^\dagger A + abz^* A^\dagger - b^2 zA^\dagger \\
&- a^2 zA^\dagger + abzA + a^2 z^* z - abz^2 \\
&+ abz^* A^\dagger - b^2 z^* A - a^2(z^*)^2 + bz^* z
\end{aligned}
$$

We calculate in the same way as we did for G^R_{11}.

$$\begin{aligned}
\frac{2}{c}G^R_{22} &= \frac{2}{c}\,\mathrm{tr}\,[\rho_0 L^R_{0,2} L^{R\,\dagger}_{0,2}] \\
&= a^2(z^* z + \frac{1}{b} + 1) - abz^2 - a^2 z^* z + abz^2 \\
&- ba(z^*)^2 + b^2(z^* z + \frac{1}{b}) + ab(z^*)^2 - b^2 z z^* \\
&- a^2 z z^* + abz^2 + a^2 z^* z - abz^2 \\
&+ ab(z^*)^2 - b^2 z^* z - a^2(z^*)^2 + bz^* z \\
&= a^2(\frac{1}{b} + 1) + b \\
&= \frac{1}{(1+2\kappa^2)^2}(1+2\kappa^2) + \frac{1}{2\kappa^2} \\
&= \frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2}
\end{aligned}$$

$$\therefore G^R_{22} = \frac{c}{2}(\frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2}) = \frac{1}{2\kappa^2(1+2\kappa^2)}\frac{1}{2}(1+4\kappa^2)c \tag{C.26}$$

G^R_{12} is

$$G^R_{12} = \mathrm{tr}\,[\rho_0 L^R_{0,2} L^{R\,\dagger}_{0,1}]$$

$$L^{2\,R}_0 = \mathrm{i}\sqrt{\frac{c}{2}}(-aA + bA^\dagger + az - bz^*) \tag{C.27}$$

We can calculate in the same way as above.

$$\begin{aligned}
\frac{2}{\mathrm{i}}G^R_{12} &= \frac{2}{\mathrm{i}}\mathrm{tr}\,[\rho_0 L^R_{0,2} L^{R\,\dagger}_{0,1}] \\
&= -a^2(z^* z + \frac{1}{b} + 1) - abz^2 + a^2 z^* z + abz^2 \\
&+ ba(z^*)^2 + b^2(z^* z + \frac{1}{b}) - ab(z^*)^2 - b^2 z z^* \\
&+ a^2 z z^* + abz^2 - a^2 z^* z - abz^2 \\
&- ab(z^*)^2 - b^2 z^* z + a^2(z^*)^2 + bz^* z \\
&= -a^2(\frac{1}{b} + 1) + b \\
&= -\frac{1}{(1+2\kappa^2)^2}(1+2\kappa^2) + \frac{1}{2\kappa^2} \\
&= -\frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2} \\
&= \frac{1}{2\kappa^2(1+2\kappa^2)}
\end{aligned}$$

$$\therefore G^R_{12} = \frac{\mathrm{i}}{2}\frac{1}{2\kappa^2(1+2\kappa^2)} \tag{C.28}$$

G_{21}^R is

$$G_{21}^R = \mathrm{tr}\,[\rho_0 L_{0,1}^R L_{0,2}^{R\dagger}]$$

From $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$,

$$\rho_0 L_{0,2}^R L_{0,1}^{R\dagger} = (L_{0,1}^R L_{0,2}^{R\dagger}\rho_0)^\dagger \tag{C.29}$$

$$\begin{aligned} G_{12}^R &= \mathrm{tr}\,[\rho_0 L_{0,2}^R L_{0,1}^{R\dagger}] \\ &= \mathrm{tr}\,[(L_{0,1}^R L_{0,2}^{R\dagger}\rho_0)^\dagger] \\ &= (\mathrm{tr}\,[L_{0,1}^R L_{0,2}^{R\dagger}\rho_0])^* \\ &= (\mathrm{tr}\,[\rho_0 L_{0,1}^R L_{0,2}^{R\dagger}])^* \\ &= (G_{21}^R)^* \end{aligned} \tag{C.30}$$

$$\therefore G_{21}^R = (G_{12}^R)^* = -\frac{\mathrm{i}}{2}\frac{1}{2\kappa^2(1+2\kappa^2)}$$

In the form of the matrix, G^R is

$$G^R = \frac{1}{2\kappa^2(1+2\kappa^2)}\frac{1}{2}\begin{pmatrix} \frac{1+4\kappa^2}{c} & \mathrm{i} \\ -\mathrm{i} & (1+4\kappa^2)c \end{pmatrix}$$

Then, the inverse of G^R, $(G^R)^{-1}$ is

$$(G^R)^{-1} = \frac{4\kappa^2(1+2\kappa^2)}{(1+4\kappa^2)^2-1}\begin{pmatrix} (1+4\kappa^2)c & -\mathrm{i} \\ \mathrm{i} & \frac{1+4\kappa^2}{c} \end{pmatrix}$$

$$(G^R)^{-1} = \frac{1}{2}\begin{pmatrix} (1+4\kappa^2)c & -\mathrm{i} \\ \mathrm{i} & \frac{1+4\kappa^2}{c} \end{pmatrix}$$

C.4 G^S for two-parameter

By the definition, SLD Fisher information G_{ij}^S is

$$G_{ij}^S = \mathrm{Re}\,(\mathrm{tr}\,[\rho L^S L^{S\dagger}]) \tag{C.31}$$

$L_0^{1\,S}$ and $L_0^{2\,S}$ are derived in the previous chapter.

$$\begin{aligned} L_{0,1}^S &= \frac{2}{c(1+4\kappa^2)}(Q - \langle Q\rangle_0) \\ L_{0,2}^S &= \frac{2c}{1+4\kappa^2}(P - \langle Q\rangle_0) \end{aligned}$$

G^S_{11} is

$$\begin{aligned} G^S_{11} &= \mathrm{Re}\,(\mathrm{tr}\,[\rho_0 L^S_{0,1} L^{S\,\dagger}_{0,1}]) \qquad \text{(C.32)} \\ \mathrm{tr}\,[\rho_0 L^S_{0,1} L^{S\,\dagger}_{0,1}] &= \frac{4}{c^2(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(Q-\langle Q\rangle_0)(Q-\langle Q\rangle_0)] \\ &= \frac{4}{c^2(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(Q^2-\langle Q\rangle_0^2)] \\ &= \frac{4}{c^2(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(Q^2-\langle Q\rangle_0^2)] \\ &= \frac{4}{c^2(1+4\kappa^2)^2}(\Delta Q)^2 \\ &= \frac{4}{c^2(1+4\kappa^2)^2}\frac{c}{2}(1+4\kappa^2) \\ &= \frac{2}{c(1+4\kappa^2)} \end{aligned}$$

$$\therefore G^S_{11} = \frac{2}{c(1+4\kappa^2)}$$

G^S_{22} is

$$G^S_{22} = \mathrm{Re}\,(\mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,2}])$$

$$\begin{aligned} \mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,2}] &= \frac{4c^2}{(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(P-\langle P\rangle_0)(P-\langle P\rangle_0)] \\ &= \frac{4c^2}{(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(P^2-\langle P\rangle_0^2)] \\ &= \frac{4c^2}{(1+4\kappa^2)^2}\mathrm{tr}\,[\rho(Q^2-\langle P\rangle_0^2)] \\ &= \frac{4c^2}{(1+4\kappa^2)^2}(\Delta Q)^2 \\ &= \frac{4c^2}{(1+4\kappa^2)^2}\frac{1}{2c}(1+4\kappa^2) \\ &= \frac{2c}{1+4\kappa^2} \\ \therefore G^S_{22} &= \frac{2c}{1+4\kappa^2} \end{aligned}$$

G^S_{12} is

$$
\begin{aligned}
G^S_{12} &= \mathrm{Re}\,(\mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,1}]) \\
\mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,1}] &= \frac{2}{c(1+4\kappa^2)} \frac{2c}{1+4\kappa^2} \mathrm{tr}\,[\rho(P - \langle P\rangle_0)(Q - \langle Q\rangle_0)] \\
&= \frac{2}{(1+4\kappa^2)^2} \mathrm{tr}\,[\rho(PQ - \langle P\rangle_0 \langle Q\rangle_0)]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}\,[\rho_0 PQ] &= \frac{1}{\mathrm{i}\sqrt{2c}}(A + A^\dagger)\sqrt{\frac{c}{2}}(A - A^\dagger) \\
&= -\frac{\mathrm{i}}{2}(A + A^\dagger)(A - A^\dagger) \\
&= -\frac{\mathrm{i}}{2}(AA - AA^\dagger + A^\dagger A - A^\dagger A^\dagger) \\
&= -\frac{\mathrm{i}}{2}(AA - 1 - A^\dagger A^\dagger) \\
&= -\frac{\mathrm{i}}{2}(z^2 - z^{*\,2} - 1) \\
\langle P\rangle_0 \langle Q\rangle_0 &= -\frac{\mathrm{i}}{2}(z + z^*)(z - z^*) \\
&= -\frac{\mathrm{i}}{2}(z^2 - z^{*\,2}) \\
\mathrm{tr}\,[\rho_0 PQ] - \langle P\rangle_0 \langle Q\rangle_0 &= \frac{\mathrm{i}}{2} \\
\mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,1}] &= \frac{\mathrm{i}}{2}\frac{2}{(1+4\kappa^2)^2} = \frac{\mathrm{i}}{(1+4\kappa^2)^2}
\end{aligned}
\tag{C.33}
$$

$$
\therefore G^S_{12} = \mathrm{Re}\,(\mathrm{tr}\,[\rho_0 L^S_{0,2} L^{S\,\dagger}_{0,1}]) = 0
$$

In the form of the matrix, G^S is

$$
G^S = \begin{pmatrix} \frac{2}{c(1+4\kappa^2)} & 0 \\ 0 & \frac{2c}{1+4\kappa^2} \end{pmatrix} = \frac{2}{1+4\kappa^2} \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & c \end{pmatrix}
$$

$$
G^S = \frac{1+4\kappa^2}{2} \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix}
$$

Then, the inverse of G^S, $(G^S)^{-1}$ is

$$
(G^S)^{-1} = \begin{pmatrix} \frac{c(1+4\kappa^2)}{2} & 0 \\ 0 & \frac{1+4\kappa^2}{2c} \end{pmatrix}
$$

Appendix D

One electron in a uniform magnetic field

D.1 Hamiltonian, annihilation and creation operators

Let $e > 0$. The charge of electron is $-e$. We choose the symmetric gauge. Then, the vector potential $\vec{A}$ for a uniform magnetic field $\vec{B} = (0, 0, B)$ is

$$\vec{A} = (-\frac{B}{2}y, \frac{B}{2}x, 0) \tag{D.1}$$

We define $\vec{\pi}$ by

$$\vec{\pi} = \vec{p} + e\vec{A} \tag{D.2}$$

Then, the Hamiltonian becomes

$$H = \frac{1}{2m}\vec{\pi}^2 = \frac{1}{2m}(\pi_1^2 + \pi_2^2 + \pi_3^2) = \frac{1}{2}(\pi_1^2 + \pi_2^2 + p_z^2) \tag{D.3}$$

Since the soltion of the z component of the wave function is a plane wave solution, we only consider the motion in x-y plane. Then, the Hamiltonian is reduced to two-dimensional.

$$H = \frac{1}{2m}(\pi_1^2 + \pi_2^2) \tag{D.4}$$

Therefore,

$$\pi_x = p_x - \frac{m\omega}{2}y$$
$$\pi_y = p_y + \frac{m\omega}{2}x$$

where $\omega = \dfrac{eB}{2}$.

The commutation relation between π_x and π_y is

$$\begin{aligned}[\pi_x, \pi_y] &= [p_x - \frac{m\omega}{2}y,\ p_y + \frac{m\omega}{2}x] \\ &= [p_x, \frac{m\omega}{2}x] - [\frac{m\omega}{2}y,\ p_y] = -\mathrm{i}m\omega\end{aligned}$$

Therefore,

$$[\frac{\pi_y}{\sqrt{m\omega}},\ \frac{\pi_x}{\sqrt{m\omega}}] = \mathrm{i}$$

It turns out to make sense if we use the replacement such as $x \to \pi_y$ and $p \to \pi_x$ in Hamiltonian of the harmonic oscillator. That is,

$$\begin{aligned}a &= \frac{\pi_y + \mathrm{i}\pi_x}{\sqrt{2m\omega}} \\ a^\dagger &= \frac{\pi_y - \mathrm{i}\pi_x}{\sqrt{2m\omega}}\end{aligned}$$

Then,

$$\begin{aligned}\pi_x &= \frac{1}{\mathrm{i}\lambda}(a - a^\dagger) \\ \pi_y &= \frac{1}{\lambda}(a + a^\dagger)\end{aligned}$$

Next, we confirm that x_0 and x_0 defined below are integrals of the motion.

$$\begin{aligned}x_0 &= x - \frac{\pi_y}{m\omega} = x - \frac{1}{m\omega}(p_y + \frac{m\omega}{2}x) \\ y_0 &= y + \frac{\pi_x}{m\omega} = y + \frac{1}{m\omega}(p_x - \frac{m\omega}{2}y)\end{aligned}$$

Then,

$$\begin{aligned}[x_0, \pi_x] &= [x - \frac{\pi_y}{m\omega},\ \pi_x] \\ &= [x,\ \pi_x] - [\frac{\pi_y}{m\omega},\ \pi_x] \\ &= [x,\ p_x] - \frac{1}{m\omega}[\pi_y,\ \pi_x] \\ &= \mathrm{i} - \frac{1}{m\omega}(\mathrm{i}m\omega) = 0\end{aligned}$$

In the same way, we have

$$[x_0, \pi_x] = [x_0, \pi_y] = 0$$

$$[y_0, \pi_x] = [y_0, \pi_y] = 0$$

$$[x_0, y_0] = \frac{\mathrm{i}}{m\omega}$$

In the same logic we define a, $a^\dagger$, if we use the replacement such as $\sqrt{m\omega}x_0 \to x$ and $\sqrt{m\omega}y_0 \to p$ in Hamiltonian of the harmonic oscillator to define anther set of operators, b, $b^\dagger$. That is,

$$\begin{aligned} b &= \sqrt{\frac{m\omega}{2}}(x_0 + \mathrm{i}y_0) = \frac{1}{\lambda}(x_0 + \mathrm{i}y_0) \\ b^\dagger &= \sqrt{\frac{m\omega}{2}}(x_0 - \mathrm{i}y_0) = \frac{1}{\lambda}(x_0 - \mathrm{i}y_0) \end{aligned}$$

Then,

$$\begin{aligned} \sqrt{m\omega}x_0 &= \frac{b + b^\dagger}{\sqrt{2}} \\ \sqrt{m\omega}y_0 &= \frac{b - b^\dagger}{\sqrt{2}\mathrm{i}} \\ x_0^2 + y_0^2 &= \frac{2}{2m\omega}(b^\dagger b + bb^\dagger) \\ &= \frac{1}{m\omega}(b^\dagger b + \frac{1}{2}) \\ &= \frac{2}{m\omega}(b^\dagger b + \frac{1}{2}) \\ &= \lambda^2(b^\dagger b + \frac{1}{2}) \end{aligned} \tag{D.5}$$

Therefore,

$$\langle n_a, n_b | x_0^2 + y_0^2 | n_a, n_b \rangle = \lambda^2(n_b + \frac{1}{2})$$

We can express x and y with using a, $a^\dagger$ and b, $b^\dagger$.

$$\begin{aligned} x &= x_0 + \frac{\pi_y}{m\omega} = \frac{\lambda}{2}(b + b^\dagger) + \frac{\lambda}{2}(a + a^\dagger) \\ y &= y_0 - \frac{\pi_x}{m\omega} = \frac{\lambda}{2\mathrm{i}}(b - b^\dagger) - \frac{\lambda}{2\mathrm{i}}(a - a^\dagger) \end{aligned}$$

p_x and p_y are

$$\begin{aligned} p_x &= \pi_x + \frac{m\omega}{2}y \\ &= \frac{1}{\mathrm{i}\lambda}(a - a^\dagger) + \frac{1}{\lambda^2}\{\frac{\lambda}{2\mathrm{i}}(b - b^\dagger) - \frac{\lambda}{2\mathrm{i}}(a - a^\dagger)\} \\ &= \frac{1}{2\mathrm{i}\lambda}(b - b^\dagger) + \frac{1}{2\mathrm{i}\lambda}(a - a^\dagger) \\ p_y &= \pi_y - \frac{m\omega}{2}x \\ &= \frac{1}{\lambda}(a + a^\dagger) - \frac{1}{\lambda^2}\{\frac{\lambda}{2}(b + b^\dagger) + \frac{\lambda}{2}(a + a^\dagger)\} \\ &= -\frac{1}{2\lambda}(b + b^\dagger) + \frac{1}{2\lambda}(a + a^\dagger) \end{aligned} \tag{D.6}$$

z component of the angular momentum l_z is

$$
\begin{aligned}
l_z &= xp_y - yp_x \\
&= \frac{1}{4}[(a + a^\dagger + b + b^\dagger)\{a + a^\dagger - (b + b^\dagger)\} + \{-(a - a^\dagger) + b - b^\dagger\}\{(a - a^\dagger) + (b - b^\dagger)\}] \\
&= \frac{1}{4}\{(a + a^\dagger)^2 - (a - a^\dagger)^2 - (b + b^\dagger)^2 + (b - b^\dagger)^2\} \\
&= \frac{1}{4}\{(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}) - (a^2 - aa^\dagger - a^\dagger a + a^{\dagger 2}) - (b^2 + bb^\dagger + b^\dagger b + b^{\dagger 2}) + (b^2 - bb^\dagger - b^\dagger b + b^{\dagger 2})\} \\
&= \frac{1}{2}\{(aa^\dagger + a^\dagger a) - (bb^\dagger + b^\dagger b)\} \\
&= \frac{1}{2}\{(2a^\dagger a + 1) - (2b^\dagger b + 1)\} \\
&= a^\dagger a - b^\dagger b
\end{aligned}
$$

$$\therefore l_z = a^\dagger a - b^\dagger b$$

D.2 Alternative representation of Unitary transformation

The unitary transformations of the model, $U_x(\theta^1), U_y(\theta^2)$ are

$$
\begin{aligned}
U_x(\theta^1) &= \mathrm{e}^{-\mathrm{i}p_x\theta^1} = \mathrm{e}^{\frac{1}{2\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}\theta^1} \\
U_y(\theta^2) &= \mathrm{e}^{-\mathrm{i}p_y\theta^2} = \mathrm{e}^{-\frac{\mathrm{i}}{2\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}\theta^2}
\end{aligned}
$$

We summarize the index of $U_x(\theta^1), U_y(\theta^2)$ by the parameters θ_1 and θ_2 so that we can see clearly the dependence of $U_x(\theta^1), U_y(\theta^2)$ on θ_1 and θ_2.

$$U_y(\theta^2)U_x(\theta^1) = \mathrm{e}^{-\frac{\mathrm{i}}{2\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}\theta^2}\mathrm{e}^{\frac{1}{2\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}\theta^1\}} \tag{D.7}$$

Since $\{(a^\dagger + a) - (b^\dagger + b)\}\theta^2$ and $\{(a^\dagger - a) + (b^\dagger - b)\}\theta^1$ commute, $U_y(\theta^2)U_x(\theta^1)$ is

$$U_y(\theta^2)U_x(\theta^1) = \mathrm{e}^{-\frac{\mathrm{i}}{2\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}\theta^2 + \frac{1}{2\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}\theta^1\}}$$

The index is

$$
\begin{aligned}
-\frac{\mathrm{i}}{2\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}\theta^2 + \frac{1}{2\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}\theta^1\} &= \frac{1}{2\lambda}\{(\theta^1 - \mathrm{i}\theta^2)a^\dagger - (\theta^1 + \mathrm{i}\theta^2)a \\
&\quad + (\theta^1 + \mathrm{i}\theta^2)b^\dagger - (\theta^1 - \mathrm{i}\theta^2)b\} \\
&= \xi a^\dagger - \xi^* a + \xi^* b^\dagger - \xi b
\end{aligned}
$$

where $\xi = \frac{1}{2\lambda}(\theta^1 - \mathrm{i}\theta^2)$.

Since $a,\ a^\dagger$ and $b,\ b^\dagger$ commute,

$$U_y(\theta^2)U_x(\theta^1) = \mathrm{e}^{\xi_a a^\dagger - \xi_a^* a}\mathrm{e}^{\xi_b b^\dagger - \xi_b^* b}$$

It is worth noting that the unitary transformation coefficient of b is complex conjugate of that of a. The unitary transformation with this form, $\mathrm{e}^{\alpha a^\dagger - \alpha^* a}$ is applied to the coherent state $|\alpha'\rangle$ is calculated as follows.

$$\begin{aligned}
U(\alpha) &= \mathrm{e}^{\alpha a^\dagger - \alpha^* a}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}\sum_{n=0}^{\infty}\frac{(-\alpha^* a)^n}{n!}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}\sum_{n=0}^{\infty}\frac{(-\alpha^* \alpha')^n}{n!}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^*\alpha'}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+2\alpha^*\alpha')}\mathrm{e}^{\alpha a^\dagger}\,|\alpha'\rangle \\
&= \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+2\alpha^*\alpha')}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\frac{1}{2}|\alpha'|^2}\mathrm{e}^{\alpha' a^\dagger}\,|0\rangle \\
&= \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2+2\alpha^*\alpha')}\mathrm{e}^{(\alpha+\alpha')a^\dagger}\,|0\rangle \\
&= \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2+2\alpha^*\alpha')}\mathrm{e}^{\frac{1}{2}|\alpha+\alpha'|^2}\mathrm{e}^{-\frac{1}{2}|\alpha+\alpha'|^2}\mathrm{e}^{(\alpha+\alpha')a^\dagger}\,|0\rangle \\
&= \mathrm{e}^{\frac{1}{2}(-\alpha^*\alpha'+\alpha\alpha'^*)}\mathrm{e}^{-\frac{1}{2}|\alpha+\alpha'|^2}\mathrm{e}^{(\alpha+\alpha')a^\dagger}\,|0\rangle \\
&= \mathrm{e}^{\frac{1}{2}(-\alpha^*\alpha'+\alpha\alpha'^*)}\,|\alpha+\alpha'\rangle \\
&= \mathrm{e}^{\mathrm{i}\Phi(\alpha,\alpha')}\,|\alpha+\alpha'\rangle \\
\therefore\ U(\alpha)\,|\alpha'\rangle\langle\alpha'|\,U(\alpha)^\dagger &= \mathrm{e}^{\mathrm{i}\Phi(\alpha,\alpha')}\,|\alpha+\alpha'\rangle\langle\alpha+\alpha'|\,\mathrm{e}^{-\mathrm{i}\Phi(\alpha,\alpha')} = |\alpha+\alpha'\rangle\langle\alpha+\alpha'|
\end{aligned} \tag{D.8}$$

D.3 Thermal state and Gaussian state

Thermal state, ρ^β and Gaussian state

$$\rho_\beta = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\,[\mathrm{e}^{-\beta H}]} \tag{D.9}$$

where $\beta = kT, k$: Boltzman constant, T temperature

When Hamittonian H is $H = \omega\,(a^\dagger a + \frac{1}{2})$, with using $a^\dagger a\,|n\rangle = n\,|n\rangle$, $e^{\beta H}$ is ,

$$e^{-\beta H} = e^{-\beta H}\sum_{n=0}^{\infty}|n\rangle\langle n| = \sum_{n=0}^{\infty}e^{-\beta H}\,|n\rangle\langle n| = \sum_{n=0}^{\infty}e^{-\beta(n\omega+\frac{1}{2})}\,|n\rangle\langle n|$$

$$Z_\beta = \mathrm{tr}\,[e^{-\beta H}] = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}e^{-\beta(n\omega+\frac{1}{2})}\,\langle k|n\rangle\langle n|k\rangle = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}e^{-\beta(n\omega+\frac{1}{2})}\delta_{k,n}\delta_{n,k} = \sum_{n=0}^{\infty}e^{-\beta(n\omega+\frac{1}{2})} = \frac{e^{-\frac{1}{2}\beta\omega}}{1-\gamma}$$

where $\gamma = e^{-\beta\omega}$

$$\therefore \rho_\beta = Z_\beta^{-1}e^{-\beta H} = (1-\gamma)\sum_n \gamma^n\,|n\rangle\langle n| \tag{D.10}$$

We first calculate the matrix element of ρ_β when the coherent state is used as a basis, $\langle z_1|\rho_\beta|z_2\rangle$. Then, we do the same on the Gaussian state to see if they match.

$\langle z_1|\rho_\beta|z_2\rangle$ is

$$\begin{aligned}
\langle z_1|\rho_\beta|z_2\rangle &= (1-\gamma)\sum_n \gamma^n\,\langle z_1|n\rangle\langle n|z_2\rangle \\
&= (1-\gamma)\sum_n \gamma^n e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\sum_{k_1}\frac{(z_1^*)^n}{\sqrt{n!}}\langle k_1|n\rangle\sum_{k_2}\frac{(z_2)^n}{\sqrt{n!}}\langle n|k_2\rangle \\
&= (1-\gamma)\sum_n \gamma^n e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\sum_{k_1}\frac{(z_1^*)^n}{\sqrt{n!}}\delta_{k_1,n}\sum_{k_2}\frac{(z_2)^n}{\sqrt{n!}}\delta_{n,k_2} \\
&= (1-\gamma)e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\sum_n \gamma^n\frac{(z_1^*z_2)^n}{n!} \\
&= (1-\gamma)e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}e^{\gamma z_1^* z_2} \\
\langle z_1|\rho^\beta|z_2\rangle &= (1-\gamma)e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2+\gamma z_1^* z_2}
\end{aligned} \tag{D.11}$$

The Gauss state S_κ is expressed as $\frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'|^2}{2\kappa^2}}\,|z'\rangle\langle z'|\,d^2z'$,

$$\begin{aligned}
\langle z_1|\,S_\kappa\,|z_2\rangle &= \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'|^2}{2\kappa^2}}\,\langle z_1|z'\rangle\langle z'|z_2\rangle\,d^2z' \\
&= \frac{1}{2\pi\kappa^2}\int e^{-(\frac{1}{2\kappa^2}+1)|z'|^2+z_1^*z_2+z_1z_2^*}d^2z'\,e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2}
\end{aligned}$$

With using

$$\int e^{-\alpha|z|^2+\beta z+\gamma z^*}d^2z = \frac{\pi}{\alpha}\exp[\frac{\beta\gamma}{\alpha}], \tag{D.12}$$

$$\langle z_1|\,\rho_\beta\,|z_2\rangle = \frac{1}{2\kappa^2+1}\exp[\frac{z_1^*z_2}{\frac{1}{2\kappa^2}+1} - \frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2] \tag{D.13}$$

The derivation of (D.12) is given at the end of this section.

To make $\langle z_1|S_\kappa|z_2\rangle = \langle z_1|\rho_\beta|z_2\rangle$,

$$
\begin{aligned}
\gamma &= \frac{1}{\frac{1}{2\kappa^2}+1} \\
1-\gamma &= \frac{1}{1+2\kappa^2} \\
\therefore 2\kappa^2 &= \frac{\gamma}{1-\gamma}
\end{aligned}
\tag{D.14}
$$

Therefore, we obtain

$$
\rho_\beta = \frac{1}{2\pi\kappa^2}\int e^{-\frac{|z'|^2}{2\kappa^2}}\,|z'\rangle\langle z'|\,d^2z' \tag{D.15}
$$

where κ is expressed as (D.14)

Derivation of (D.12)

$$
\int e^{-\alpha|z|^2+\beta z+\gamma z^*}d^2z = \frac{\pi}{\alpha}\exp[\frac{\beta\gamma}{\alpha}],
$$

where $\alpha > 0$ and $\beta,\ \gamma \in \mathbf{C}$.

Let $z = x + \mathrm{i}y$,

$$
\int e^{-\alpha|z|^2+\beta z+\gamma z^*}d^2z = \int e^{-\alpha(x^2+y^2)+\beta(x+\mathrm{i}y)+\gamma(x-\mathrm{i}y)}d^2z = \int_{-\infty}^{\infty} e^{-\alpha x^2+(\beta+\gamma)x}dx\int_{-\infty}^{\infty} e^{-\alpha y^2+\mathrm{i}(\beta-\gamma)y}dy \tag{D.16}
$$

We execute the first integration.

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\alpha x^2+(\beta+\gamma)x}dx &= \int_{-\infty}^{\infty} e^{-\alpha(x^2-\frac{\beta+\gamma}{\alpha}x)}dx \\
&= \int_{-\infty}^{\infty} e^{-\alpha(x-\frac{\beta+\gamma}{2\alpha})^2+\alpha\frac{(\beta+\gamma)^2}{4\alpha^2}}dx \\
&= \int_{-\infty}^{\infty} e^{-\alpha(x-\frac{\beta+\gamma}{2\alpha})^2}dx\,\mathrm{e}^{\frac{(\beta+\gamma)^2}{4\alpha}}
\end{aligned}
$$

Let $t = x - \frac{\beta+\gamma}{2\alpha}$,

$$
\int_{-\infty}^{\infty} e^{-\alpha(x-\frac{\beta+\gamma}{2\alpha})^2}dx\,\mathrm{e}^{\frac{(\beta+\gamma)^2}{4\alpha}} = \mathrm{e}^{\frac{(\beta+\gamma)^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha t^2}dt = \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{\frac{(\beta+\gamma)^2}{4\alpha}} \tag{D.17}
$$

Then, the second term is

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\alpha y^2-\mathrm{i}(\beta-\gamma)y}dy &= \int_{-\infty}^{\infty} e^{-\alpha(y^2-\mathrm{i}\frac{\beta-\gamma}{\alpha}y)}dy \\
&= \int_{-\infty}^{\infty} e^{-\alpha\{y^2-\mathrm{i}\frac{\beta-\gamma}{\alpha}y+\frac{(\beta-\gamma)^2}{4\alpha^2}-\frac{(\beta-\gamma)^2}{4\alpha^2}\}}dy \\
&= \int_{-\infty}^{\infty} e^{-\alpha\{y-\mathrm{i}\frac{\beta-\gamma}{2\alpha}\}^2}dy\,\mathrm{e}^{-\frac{(\beta-\gamma)^2}{4\alpha}} \\
&= \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{-\frac{(\beta-\gamma)^2}{4\alpha}}
\end{aligned}
$$

Multiplying the first term by the second term gives the integration we need.

$$\int e^{-\alpha|z|^2+\beta z+\gamma z^*} d^2z = \sqrt{\frac{\pi}{\alpha}} \mathrm{e}^{\frac{(\beta+\gamma)^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}} \mathrm{e}^{-\frac{(\beta-\gamma)^2}{4\alpha}} = \frac{\pi}{\alpha} \mathrm{e}^{\frac{4\beta\gamma}{4\alpha}} = \frac{\pi}{\alpha} \mathrm{e}^{\frac{\beta\gamma}{\alpha}} \tag{D.18}$$

D.4 Construction of the thermal state with the assumption of the constant angular momentum

As shown in (D.1), l_z is expressed as

$$l_z = a^\dagger a - b^\dagger b. \tag{D.19}$$

$$\rho_{\beta,\mu} = \frac{\mathrm{e}^{-\beta H+\mu l_z}}{\mathrm{tr}[\mathrm{e}^{-\beta H+\mu l_z}]} = \frac{\mathrm{e}^{-\beta\omega a^\dagger a+\mu(a^\dagger a-b^\dagger b)}}{Z_{\beta,\mu}} = \frac{\mathrm{e}^{-(\beta\omega-\mu)a^\dagger a-\mu b^\dagger b}}{Z_{\beta,\mu}} \tag{D.20}$$

$Z_{\beta,\mu} = \mathrm{tr}[\mathrm{e}^{-(\beta\omega-\mu)a^\dagger a-\mu b^\dagger b}]$

With using,

$$\begin{aligned} a^\dagger a \left|n\right\rangle_a &= n_a \left|n\right\rangle_a \\ b^\dagger b \left|n\right\rangle_b &= n_b \left|n\right\rangle_b \end{aligned} \tag{D.21}$$

$$\rho_{\beta,\mu} = \frac{1}{Z_{\beta,\mu}} \sum_n \sum_i \mathrm{e}^{-(\beta\omega-\mu)a^\dagger a-\mu b^\dagger b} \left|n\right\rangle_a \otimes \left|i\right\rangle_b \, {}_a\!\left\langle n\right| \otimes_b \left\langle i\right| \tag{D.22}$$

$$= \frac{1}{Z_{\beta,\mu}} \sum_n \mathrm{e}^{-(\beta\omega-\mu)a^\dagger a} \left|n\right\rangle_a \left\langle n\right| \sum_i \mathrm{e}^{-\mu b^\dagger b} \left|i\right\rangle_b \left\langle i\right| \tag{D.23}$$

$$= \frac{1}{Z_{\beta,\mu}} \{\sum_n \mathrm{e}^{-(\beta\omega-\mu)n} \left|n\right\rangle_a \left\langle n\right| \otimes I_b\}\{I_a \otimes \sum_i \mathrm{e}^{-\mu i} \left|i\right\rangle_b \left\langle i\right|\} \tag{D.24}$$

Therefore, ρ_0 is in the form of $\rho_{0,a} \otimes \rho_{0,b}$. Then, $\rho_{0,a}$ and $\rho_{0,b}$ are

$$\rho_{0,a} = \frac{1}{2\pi\kappa_a^2} \int e^{-\frac{|z|^2}{2\kappa_a^2}} \left|z\right\rangle \left\langle z\right| d^2z \tag{D.25}$$

$$\rho_{0,b} = \frac{1}{2\pi\kappa_b^2} \int e^{-\frac{|z|^2}{2\kappa_b^2}} \left|z\right\rangle \left\langle z\right| d^2z \tag{D.26}$$

$$\tag{D.27}$$

In the same way as we derived (D.15), we obtain γ_a and γ_b as follows.

$$\gamma_a = \mathrm{e}^{-(\beta\omega-\mu)}$$

$$\gamma_b = \mathrm{e}^{-\mu}$$

We next calculate $Z_{\beta,\mu}$.

$$Z_{\beta,\mu} = \mathrm{tr}\,[\mathrm{e}^{-(\beta\omega-\mu)a^\dagger a-\mu b^\dagger b}] \tag{D.28}$$
$$= \sum_n {}_a\langle n|\,\mathrm{e}^{-(\beta\omega-\mu)n}\,|n\rangle_a \sum_i {}_b\langle i|\,\mathrm{e}^{-\mu i}\,|i\rangle_b \tag{D.29}$$
$$= \sum_n \mathrm{e}^{-(\beta\omega-\mu)n} \sum_i \mathrm{e}^{-\mu i} \tag{D.30}$$
$$= \sum_n \gamma_a^n \sum_i \gamma_a^i \tag{D.31}$$
$$= \frac{1}{1-\gamma_a}\frac{1}{1-\gamma_b} \tag{D.32}$$
$$\rho_{\beta,\mu} = \frac{1}{1-\gamma_a}\{\sum_n \gamma_a^n\,|n\rangle_a\,\langle n|\}\{\frac{1}{1-\gamma_b}\sum_i \gamma_b^i\,|i\rangle_b\,\langle i|\} \tag{D.33}$$

We can summarize the results as follows.

$$\rho_{\beta,\mu} = \rho_a \otimes \rho_b \tag{D.34}$$
$$\rho_a = \frac{1}{2\pi\kappa_b^2}\int e^{-\frac{|z|^2}{2\kappa_b^2}}\,|z\rangle\,\langle z|\,d^2z \tag{D.35}$$
$$\rho_b = \frac{1}{2\pi\kappa_a^2}\int e^{-\frac{|z|^2}{2\kappa_a^2}}\,|z\rangle\,\langle z|\,d^2z \tag{D.36}$$
$$2\kappa_a^2 = \frac{\gamma_a}{1-\gamma_a} \tag{D.37}$$
$$2\kappa_b^2 = \frac{\gamma_b}{1-\gamma_b} \tag{D.38}$$
$$\tag{D.39}$$

Next, we derive the condition the μ should satisfy. The expectation value of $l_z = \langle l_z\rangle$ is

$$l_z = a^\dagger a - b^\dagger b$$
$$\langle l_z\rangle = \mathrm{tr}[\rho_a \otimes \rho_b(a^\dagger a - b^\dagger b)]$$
$$= \mathrm{tr}[\rho_a(a^\dagger a)] - \mathrm{tr}[\rho_b(b^\dagger b)]$$
$$= 2\kappa_a^2 - 2\kappa_b^2 \tag{D.40}$$

$$\langle l_z \rangle = 2\kappa_a^2 - 2\kappa_b^2 \tag{D.41}$$

$$2\kappa_b^2 = 2\kappa_a^2 + \langle l_z \rangle \tag{D.42}$$

$$2\kappa_b^2 = \frac{\gamma_b}{1-\gamma_b} = \frac{1}{e^{\mu}-1} \tag{D.43}$$

$$e^{\mu} = 1 + \frac{1}{2\kappa_a^2} \tag{D.44}$$

$$e^{-\mu} = \frac{2\kappa_b^2}{1+2\kappa_b^2} = \frac{2\kappa_a^2 + \langle l_z \rangle}{1+2\kappa_a^2 + \langle l_z \rangle} \tag{D.45}$$

$$\frac{1}{2\kappa_a^2} = e^{\beta\omega-\mu} - 1 = e^{\beta\omega}\frac{2\kappa_a^2 + \langle l_z \rangle}{1+2\kappa_a^2 + \langle l_z \rangle} - 1 \tag{D.46}$$

Let $s = 2\kappa_a^2$

$$\{(e^{\beta\omega}-1)s + (e^{\beta\omega}-1)\langle l_z \rangle - 1\}s = 1 + s + \langle l_z \rangle \tag{D.47}$$

$$(e^{\beta\omega}-1)s^2 + \{(e^{\beta\omega}-1)\langle l_z \rangle - 1\}s - s = 1 + \langle l_z \rangle \tag{D.48}$$

$$(e^{\beta\omega}-1)s^2 + \{(e^{\beta\omega}-1)\langle l_z \rangle - 2\}s - 1 - \langle l_z \rangle = 0 \tag{D.49}$$

Let $t = e^{\beta\omega} - 1$

$$ts^2 + (t\langle l_z \rangle - 2)s - 1 - \langle l_z \rangle = 0$$

We analyze the solutions of the equation above to find out the relationship between μ and $\langle l_z \rangle$. Let $u(s) = ts^2 + (t\langle l_z \rangle - 2)s - 1 - \langle l_z \rangle$. From $t > 0$, $u(s)$ is downward convex. s_0 defined as the solution of $\frac{du(s_0)}{ds} = 0$ is $s_0 = \frac{-t\langle l_z \rangle + 2}{2t}$

① When $s_0 > 0 \rightarrow -t\langle l_z \rangle + 2 > 0 \land u(0) > 0$, two positive solutions exist.

$$\langle l_z \rangle < \frac{2}{t} \ \land \ \langle l_z \rangle < -1 \rightarrow \langle l_z \rangle < -1 \tag{D.50}$$

② When $s_0 > 0 \rightarrow -t\langle l_z \rangle + 2 > 0 \land u(0) \leq 0$, one positive solution exits.

$$\langle l_z \rangle < \frac{2}{t} \ \land \ \langle l_z \rangle \geq -1 \tag{D.51}$$

③ When $s_0 < 0 \rightarrow -t\langle l_z \rangle + 2 \leq 0 \land u(0) < 0$, one positive solution exits.

$$\langle l_z \rangle > \frac{2}{t} \ \land \ \langle l_z \rangle \geq -1 \tag{D.52}$$

④ When $s_0 < 0 \rightarrow -t\langle l_z \rangle + 2 < 0 \land u(0) > 0$, no positive solution exists.

$$\langle l_z \rangle > \frac{2}{t} \ \land \ \langle l_z \rangle < -1 \tag{D.53}$$

Since $t > 0$, the region that satisfies ④ does not exist.

In the case of ①

The smaller solution is calculated as

$$s = 2\kappa_a^2 = \frac{1}{2t}\{-(t\langle l_z\rangle - 2) - \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} \tag{D.54}$$

From $\langle l_z\rangle < -1$, $-(t\langle l_z\rangle - 2) > t + 2 > 0$ and $2\kappa_b^2 = 2\kappa_a^2 + \langle l_z\rangle$, we have

$$\begin{aligned} 2\kappa_b^2 &= \frac{1}{2t}\{-(t\langle l_z\rangle - 2) - \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} + \langle l_z\rangle \\ &= \frac{1}{2t}\{t\langle l_z\rangle + 2 - \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} \end{aligned} \tag{D.55}$$

From $2\kappa_b^2 > 0 \wedge t > 0$,

$$\begin{aligned} t\langle l_z\rangle + 2 - \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)} &> 0 \\ t\langle l_z\rangle + 2 - \sqrt{\langle l_z\rangle^2 t^2 + 4t + 4} &> 0 \\ t\langle l_z\rangle + 2 - \sqrt{\langle l_z\rangle^2 t^2 + 4t + 4} &> 0 \\ (t\langle l_z\rangle + 2)^2 &> \langle l_z\rangle^2 t^2 + 4t + 4 \\ \langle l_z\rangle^2 t^2 + 4\langle l_z\rangle t + 4 &> \langle l_z\rangle^2 t^2 + 4t + 4 \\ \langle l_z\rangle &> 1 \end{aligned}$$

When $\langle l_z\rangle < -1$, $2\kappa_b^2 < 0$. Therefore, we do not accept the smaller solution in the case ① as our solution, because the solution gives negative $2\kappa_b^2$.

$2\kappa_b^2$ we obtain from the larger solution is

$$\begin{aligned} 2\kappa_b^2 &= 2\kappa_a^2 + \langle l_z\rangle \\ 2\kappa_b^2 &= \frac{1}{2t}\{-(t\langle l_z\rangle - 2) + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} + \langle l_z\rangle \\ &= \frac{1}{2t}\{t\langle l_z\rangle + 2 + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} \end{aligned} \tag{D.56}$$

Again, we use $2\kappa_b^2 > 0 \wedge t > 0$.

$$
\begin{aligned}
t\langle l_z\rangle + 2 + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)} &> 0 \\
t\langle l_z\rangle + 2 + \sqrt{\langle l_z\rangle^2 t^2 + 4t + 4} &> 0 \\
t\langle l_z\rangle + 2 &> -\sqrt{\langle l_z\rangle^2 t^2 + 4t + 4} \\
-(t\langle l_z\rangle + 2) &< \sqrt{\langle l_z\rangle^2 t^2 + 4t + 4} \\
(t\langle l_z\rangle + 2)^2 &< \langle l_z\rangle^2 t^2 + 4t + 4 \\
\langle l_z\rangle^2 t^2 + 4\langle l_z\rangle t + 4 &< \langle l_z\rangle^2 t^2 + 4t + 4 \\
\langle l_z\rangle &< 1
\end{aligned}
$$

Since $\langle l_z\rangle < -1$ is confirmed, we take the larger solution in the case ①.

We calculate μ from the solution.

$$
\begin{aligned}
2\kappa_b^2 &= 2\kappa_a^2 + \langle l_z\rangle \\
2\kappa_b^2 &= \frac{1}{2t}\{-(t\langle l_z\rangle - 2) + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} + \langle l_z\rangle \\
\frac{1}{\mathrm{e}^{-\mu} - 1} &= \frac{1}{2t}\{t\langle l_z\rangle + 2 + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}\} \qquad \text{(D.57)} \\
\mathrm{e}^{-\mu} &= \frac{2t}{t\langle l_z\rangle + 2 + \sqrt{(t\langle l_z\rangle - 2)^2 + 4t(1 + \langle l_z\rangle)}} + 1 \qquad \text{(D.58)}
\end{aligned}
$$

D.5 RLD and SLD Fisher information matrix: Model 1

D.5.1 RLD Fisher information matrix G^R

We choose the thermal state as the reference state.

$$
\rho_0 = \frac{1}{2\pi\kappa^2}\int \mathrm{e}^{\frac{-|z|^2}{2}} |z\rangle\langle z|\, d^2 z
$$

With a, $a^\dagger$, π_x and π_y are expressed as

$$
\begin{aligned}
\pi_x &= \frac{1}{\mathrm{i}\lambda}(a - a^\dagger) \\
\pi_y &= \frac{1}{\lambda}(a + a^\dagger)
\end{aligned}
$$

ρ_θ and $\dfrac{\partial\rho(\theta)}{\partial\theta^1}$ are

$$
\begin{aligned}
\rho_\theta &= \mathrm{e}^{-\mathrm{i}\pi_x\theta^1}\mathrm{e}^{-\mathrm{i}\pi_y\theta^2}\rho_0(\mathrm{e}^{-\mathrm{i}\pi_x\theta^1}\mathrm{e}^{-\mathrm{i}\pi_y\theta^2})^\dagger = \mathrm{e}^{-\mathrm{i}\pi_x\theta^1}\mathrm{e}^{-\mathrm{i}\pi_y\theta^2}\rho_0\mathrm{e}^{\mathrm{i}\pi_y\theta^2}\mathrm{e}^{\mathrm{i}\pi_x\theta^1} \\
\frac{\partial\rho_\theta}{\partial\theta^1} &= -\mathrm{i}\pi_x\rho_\theta + i\rho_\theta\,\pi_x \\
&= -\frac{1}{\lambda}(a-a^\dagger)\rho_\theta + \rho_\theta\frac{1}{\lambda}(a-a^\dagger) \\
&= -\frac{1}{\lambda}(a\rho_\theta - a\rho_\theta) + \frac{1}{\lambda}(a^\dagger\rho_\theta - \rho_\theta a^\dagger) \\
&= -\frac{1}{\lambda}[a,\rho_\theta] + \frac{1}{\lambda}[a^\dagger,\rho_\theta] \\
&= \frac{1}{\lambda}[a^\dagger - a,\, \rho_\theta] \\
&= U(\theta^1,\theta^2)\frac{1}{\lambda}[a^\dagger - a,\, \rho_0]U^\dagger(\theta^1,\theta^2)
\end{aligned}
$$

By using the commutation relations given in Appendix B.3

$$
\begin{aligned}
a^\dagger\rho_0 &= \frac{1+2\kappa^2}{2\kappa^2}\rho_0\, a^\dagger \\
a\rho_0 &= \frac{2\kappa^2}{1+2\kappa^2}\rho_0\, a \\
[a,\rho_0] &= a\rho_0 - \rho_0 a = (\frac{2\kappa^2}{1+2\kappa^2} - 1)\rho_0 a = -\frac{1}{1+2\kappa^2}\rho_0 a \\
[a^\dagger,\rho_0] &= a^\dagger\rho_0 - \rho_0 a^\dagger = (\frac{1+2\kappa^2}{2\kappa^2} - 1)\rho_0 a^\dagger = \frac{1}{2\kappa^2}\rho_0 a^\dagger
\end{aligned}
\tag{D.59}
$$

Therefore $[a^\dagger - a,\, \rho_0]$ is

$$
\begin{aligned}
[a^\dagger - a,\, \rho_0] &= \rho_0(\frac{1}{2\kappa^2}a^\dagger + \frac{1}{1+2\kappa^2}a) \\
\therefore\ \frac{\partial\rho_\theta}{\partial\theta^1} &= U(\theta^1,\theta^2)\,\rho_0(\frac{1}{2\kappa^2}a^\dagger + \frac{1}{1+2\kappa^2}a)U^\dagger(\theta^1,\theta^2)
\end{aligned}
$$

By comparing these relations with the definition of the RLD,

$$
\frac{\partial\rho_\theta}{\partial\theta^i} = \rho_\theta L^R_{\theta\,i}, \tag{D.60}
$$

we have

$$
L^R_{0\,1} = \frac{1}{\lambda}(\frac{1}{1+2\kappa^2}a + \frac{1}{2\kappa^2}a^\dagger), \tag{D.61}
$$

where $L^R_{\theta\,i} = U(\theta^1,\theta^2)\, L^R_{0\,1}U^\dagger(\theta^1,\theta^2)$ and $U(\theta^1,\theta^2) = \mathrm{e}^{-\mathrm{i}\pi_x\theta^1}\mathrm{e}^{-\mathrm{i}\pi_y\theta^2}$.

The differentiation of ρ_θ with respect to θ^2 is

$$\begin{aligned}\frac{\partial\rho_\theta}{\partial\theta^2} &= -\mathrm{i}\pi_y\rho_\theta + i\rho_\theta\pi_y \\ &= U(\theta^1,\theta^2)\frac{(-\mathrm{i})}{\lambda}[a^\dagger + a,\, \rho_0]U^\dagger(\theta^1,\theta^2)\end{aligned}$$

$[a^\dagger + a,\, \rho_0]$ is

$$[a^\dagger + a,\, \rho_0] = \rho_0(\frac{1}{2\kappa^2}a^\dagger - \frac{1}{1+2\kappa^2}a)$$

Therefore,

$$L^R_{0\,2} = \frac{-\mathrm{i}}{\lambda}(-\frac{1}{1+2\kappa^2}a + \frac{1}{2\kappa^2}a^\dagger) = \frac{\mathrm{i}}{\lambda}(\frac{1}{1+2\kappa^2}a - \frac{1}{2\kappa^2}a^\dagger). \tag{D.62}$$

Then, we can calculate RLD Fisher information by

$$[G^R(\theta)]_{i\,j} = g^R_{i\,j} = \mathrm{tr}[\rho_\theta L^R_{\theta\,j}L^R_{\theta\,i}] = \mathrm{tr}\,[\rho_0\, L^R_{0\,j}L^R_{0\,i}]. \tag{D.63}$$

Calculation of $\mathrm{tr}\,[\rho_0\, L^R_{0\,j}L^R_{0\,i}]$ is shown below. Let $\alpha = \frac{1}{1+2\kappa^2}$ and $\beta = \frac{1}{2\kappa^2}$. Then,

$$L^R_{0\,2} = \frac{\mathrm{i}}{\lambda}(\alpha a - \beta a^\dagger) \tag{D.64}$$

$$\begin{aligned}g^R_{1\,1} = \mathrm{tr}\,[\rho_0\, L^R_{0\,1}L^{R\,\dagger}_{0\,1}] &= \mathrm{tr}\,[\rho_0\,\frac{1}{\lambda^2}(\alpha a + \beta a^\dagger)(\alpha a^\dagger + \beta a)] \\ &= \frac{1}{\lambda^2}\mathrm{tr}\,[\rho_0\,(\alpha^2 aa^\dagger + \alpha\beta a^2 + \beta\alpha(a^\dagger)^2 + \beta^2 a^\dagger a)]\end{aligned}$$

From $\mathrm{tr}\,[\rho_0\, a^2] = \mathrm{tr}\,[\rho_0\,(a^\dagger)^2] = 0$,

$$\begin{aligned}g^R_{1\,1} = \mathrm{tr}\,[\rho_0\, L^R_{0\,1}L^{R\,\dagger}_{0\,1}] &= \frac{1}{\lambda^2}\mathrm{tr}\,[\rho_0\,\{\alpha^2(a^\dagger a + 1) + \beta^2 a^\dagger a\}] \\ &= \frac{1}{\lambda^2}\mathrm{tr}\,[\rho_0\,\{\alpha^2(a^\dagger a + 1) + \beta^2 a^\dagger a\}] \\ &= \frac{1}{\lambda^2}\{\frac{1+2\kappa^2}{(1+2\kappa^2)^2} + \frac{2\kappa^2}{(2\kappa^2)^2}\}] \\ &= \frac{1}{\lambda^2}(\frac{1}{1+2\kappa^2} + \frac{1}{2\kappa^2}) \\ &= \frac{1}{\lambda^2}\frac{1+4\kappa^2}{2\kappa^2(1+2\kappa^2)}\end{aligned}$$

$$\begin{aligned}g^R_{1\,2} = \mathrm{tr}\,[\rho_0\, L^R_{0\,2}L^{R\,\dagger}_{0\,1}] &= \mathrm{tr}\,[\rho_0\,\frac{\mathrm{i}}{\lambda^2}(\alpha a - \beta a^\dagger)(\alpha a^\dagger + \beta a)] \\ &= \mathrm{tr}\,[\rho_0\,\frac{\mathrm{i}}{\lambda^2}(\alpha^2 aa^\dagger + \alpha\beta a^2 - \beta\alpha(a^\dagger)^2 - \beta^2 a^\dagger a)] \\ &= \mathrm{tr}\,[\rho_0\,\frac{\mathrm{i}}{\lambda^2}\{\alpha^2(a^\dagger a + 1) + \alpha\beta a^2 - \beta\alpha(a^\dagger)^2 - \beta^2 a^\dagger a\}] \\ &= \frac{\mathrm{i}}{\lambda^2}(\frac{1}{1+2\kappa^2} - \frac{1}{2\kappa^2}) \\ &= -\frac{\mathrm{i}}{\lambda^2}\frac{1}{2\kappa^2(1+2\kappa^2)}\end{aligned}$$

Therefore, RLD Fisher information matrix and its inverse are

$$G^R(\theta) = \frac{1}{\lambda^2}\frac{1}{2\kappa^2(1+2\kappa^2)}\begin{pmatrix} 1+4\kappa^2 & -\mathrm{i} \\ \mathrm{i} & 1+4\kappa^2 \end{pmatrix}$$

$$(G^R(\theta))^{-1} = \frac{2\lambda^2\kappa^2(1+2\kappa^2)}{(1+4\kappa^2)^2-1}\begin{pmatrix} 1+4\kappa^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa^2 \end{pmatrix} = \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa^2 \end{pmatrix}$$

D.5.2 SLD Fisher information matrix G^S

Again, $\dfrac{\partial\rho(\theta)}{\partial\theta^1}$ is

$$\frac{\partial\rho(\theta)}{\partial\theta^1} = \frac{1}{\lambda}[a^\dagger - a,\, \rho_\theta]$$

We can derive the relation between $[a^\dagger - a,\, \rho_0]$ and $\{a^\dagger + a,\, \rho_0\}$ as follows.

$$\begin{aligned}
\{a + a^\dagger,\, \rho_0\} &= \{a,\, \rho_0\} + \{a^\dagger,\, \rho_0\} \\
&= (a\,\rho_0 + \rho_0\, a) + (a^\dagger\,\rho_0 + \rho_0\, a^\dagger) \\
&= (\frac{2\kappa^2}{1+2\kappa^2} + 1)\,\rho_0 a + (\frac{1+2\kappa^2}{2\kappa^2} + 1)\,\rho_0 a^\dagger \\
&= \frac{1+4\kappa^2}{1+2\kappa^2}\,\rho_0 a + \frac{1+4\kappa^2}{2\kappa^2}\,\rho_0 a^\dagger \\
&= (1+4\kappa^2)\,\rho_0\,(\frac{1}{1+2\kappa^2}\,a + \frac{1}{2\kappa^2}a^\dagger)
\end{aligned}$$

$$\therefore\ [a^\dagger - a,\, \rho_0] = \rho_0\,(\frac{1}{2\kappa^2}a^\dagger + \frac{1}{1+2\kappa^2}a) = \frac{1}{1+4\kappa^2}\,\rho_0\,\{a^\dagger + a,\, \rho_0\} \tag{D.65}$$

$$\frac{\partial\rho(\theta)}{\partial\theta^1} = U(\theta^1,\theta^2)\,\frac{1}{2}\frac{1}{\lambda}\frac{2}{1+4\kappa^2}\,\rho_0\,\{a^\dagger + a,\, \rho_0\}U^\dagger(\theta^1,\theta^2)$$

By comparing the equation above with the definition of the SLD,

$$\frac{\partial\rho_\theta}{\partial\theta^i} = \frac{1}{2}(L^S_{\theta\, i}\rho_\theta + \rho_\theta L^S_{\theta\, i}),$$

we obtatin

$$L^S_{0\,1} = \frac{2}{\lambda(1+4\kappa^2)}(a + a^\dagger). \tag{D.66}$$

where $L^S_{\theta\, i} = U(\theta^1,\theta^2)\, L^S_{0\,1}U^\dagger(\theta^1,\theta^2)$.

The differentiation of ρ_θ with respect to θ^2 is

$$\begin{aligned}
\frac{\partial\rho_\theta}{\partial\theta^2} &= -\mathrm{i}\pi_y\rho_\theta + i\rho_\theta\,\pi_y \\
&= U(\theta^1,\theta^2)\frac{1}{2}\frac{(-2\mathrm{i})}{\lambda}[a^\dagger + a,\, \rho_0]U^\dagger(\theta^1,\theta^2).
\end{aligned}$$

$$
\begin{aligned}
\{a - a^\dagger,\, \rho_0\} &= \{a,\, \rho_0\} - \{a^\dagger,\, \rho_0\} \\
&= (a\,\rho_0 + \rho_0\, a) - (a^\dagger\, \rho_0 + \rho_0\, a^\dagger) \\
&= (\frac{2\kappa^2}{1+2\kappa^2} + 1)\,\rho_0 a - (\frac{1+2\kappa^2}{2\kappa^2} + 1)\,\rho_0 a^\dagger \\
&= \frac{1+4\kappa^2}{1+2\kappa^2}\,\rho_0 a - \frac{1+4\kappa^2}{2\kappa^2}\,\rho_0 a^\dagger \\
&= (1+4\kappa^2)\,\rho_0\,(\frac{1}{1+2\kappa^2}\,a - \frac{1}{2\kappa^2} a^\dagger)
\end{aligned}
$$

$$
\therefore\ [a^\dagger + a,\, \rho_0] = \rho_0\,(\frac{1}{2\kappa^2} a^\dagger - \frac{1}{1+2\kappa^2} a) = -\frac{1}{1+4\kappa^2}\,\rho_0\,\{a - a^\dagger,\, \rho_0\} \tag{D.67}
$$

$$
\frac{\partial\rho(\theta)}{\partial\theta^2} = U(\theta^1, \theta^2)\,\frac{1}{2}\frac{\mathrm{i}}{\lambda}\frac{2}{1+4\kappa^2}\,\rho_0\,\{a - a^\dagger,\, \rho_0\} U^\dagger(\theta^1, \theta^2)
$$

In the same way as above, we obtain

$$
L^S_{0\,2} = \frac{2\mathrm{i}}{\lambda(1+4\kappa^2)}(a - a^\dagger) \tag{D.68}
$$

Then, we can calculate SLD Fisher information by

$$
[G^R(\theta)]_{i\,j} = g^R_{i\,j} = \mathrm{Re}\,\mathrm{tr}[\rho_\theta L^S_{\theta\,j} L^S_{\theta\,i}] = \mathrm{Re}\,\mathrm{tr}[\rho_0\, L^S_{0\,j} L^S_{0\,i}]. \tag{D.69}
$$

Calculation of $\mathrm{tr}[\rho_0\, L^S_{0\,j} L^S_{0\,i}]$ is shown below.

$$
\begin{aligned}
\mathrm{tr}\,[\rho_0\, L^S_{0\,1} L^S_{0\,1}] &= \mathrm{tr}\,[\rho_0\, \frac{4}{\lambda^2(1+4\kappa^2)^2}(a + a^\dagger)^2] \\
&= \frac{4}{\lambda^2(1+4\kappa^2)^2}\,\mathrm{tr}\,[\rho_0\,(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)] \\
&= \frac{4}{\lambda^2(1+4\kappa^2)^2}\,\mathrm{tr}\,[\rho_0\,(a^2 + 2a^\dagger a + 1 + (a^\dagger)^2)] \\
&= \frac{4}{\lambda^2(1+4\kappa^2)^2}(1+4\kappa^2) \\
&= \frac{4}{\lambda^2(1+4\kappa^2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}[\rho_0\, L^S_{0\,2} L^S_{0\,1}] &= \mathrm{tr}\,[\rho_0\, \frac{4\mathrm{i}}{\lambda^2(1+4\kappa^2)^2}(a - a^\dagger)(a + a^\dagger)] \\
&= \frac{4\mathrm{i}}{\lambda^2(1+4\kappa^2)^2}\,\mathrm{tr}\,[\rho_0\,(a^2 + aa^\dagger - a^\dagger a - (a^\dagger)^2)] \\
&= \frac{4\mathrm{i}}{\lambda^2(1+4\kappa^2)^2}\,\mathrm{tr}\,[\rho_0\,(a^2 + 1 - (a^\dagger)^2)] \\
&= \frac{4\mathrm{i}}{\lambda^2(1+4\kappa^2)^2}
\end{aligned}
$$

Therefore, SLD Fisher information matrix and its inverse are

$$G^S(\theta) = \frac{4}{\lambda^2(1+4\kappa^2)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$(G^S(\theta))^{-1} = \frac{\lambda^2(1+4\kappa^2)}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\tilde{G^S}(\theta) = \frac{4}{\lambda^2(1+4\kappa^2)}\begin{pmatrix}1 & \frac{\mathrm{i}}{1+4\kappa^2}\\ -\frac{\mathrm{i}}{1+4\kappa^2} & 1\end{pmatrix}$$

where

$$[\tilde{G^S}(\theta)]_{ij} = \mathrm{tr}\,[\rho_0\, L^S_{0\,j} L^S_{0\,i}]$$

We can also calculate $Z(\theta)$ matrix as follows.

$$Z(\theta) = (G^S(\theta))^{-1}\tilde{G^S}(\theta)(G^S(\theta))^{-1}$$

$$\begin{aligned}
Z(\theta) &= (G^S(\theta))^{-1}(G^S(\theta) + \frac{1}{\lambda^2(1+4\kappa^2)^2}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix})(G^S(\theta))^{-1}\\
&= (G^S(\theta))^{-1} + \frac{4}{\lambda^2(1+4\kappa^2)^2}\frac{\lambda^4(1+4\kappa^2)^2}{16}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\\
&= \frac{\lambda^2}{4}\begin{pmatrix}1+4\kappa^2 & \mathrm{i}\\ -\mathrm{i} & 1+4\kappa^2\end{pmatrix}\\
\therefore Z(\theta) &= (G^R(\theta))^{-1} \qquad \text{(D.70)}
\end{aligned}$$

Therefore, Model 1 is D-invariant and $(G^R(\theta))^{-1}$ gives achievable bound.

D.6 RLD and SLD Fisher information: Model 2

We choose the thermal state as the reference state. Therefore,

$$\rho_0 = \rho_a \otimes \rho_b. \qquad \text{(D.71)}$$

where

$$\begin{aligned}
\rho_{0\,a} &= \frac{1}{2\pi\kappa_a^2}\int e^{-\frac{|z|^2}{2\kappa^2}}\,|z\rangle_a\langle z|\,d^2z\\
\rho_{0\,b} &= \frac{1}{2\pi\kappa_a^2}\int e^{-\frac{|z|^2}{2\kappa^2}}\,|z\rangle_b\langle z|\,d^2z
\end{aligned} \qquad \text{(D.72)}$$

Since $\rho_0 = \rho_a \otimes \rho_b$ and since $a, a^\dagger$ and $b, b^\dagger$ satisfy the same commutation relation, $[a, a^\dagger] = [b, b^\dagger] = 1$, we regard $a, a^\dagger$ and $b, b^\dagger$ as the same operator set, but act on the two different Hilbert space, $\mathscr{H}_a$ and $\mathscr{H}_b$. Therefore, we use the notation below hereafter.

$$
\begin{aligned}
a &\to a \otimes I_b \\
a^\dagger &\to a \otimes I_b \\
b &\to I_a \otimes a \\
b^\dagger &\to I_a \otimes a^\dagger
\end{aligned}
$$

As shown in Appendix B.3, we have these commutation relations.

$$
\begin{aligned}
a^\dagger S_\kappa &= \frac{1}{2\kappa^2}\{(2\kappa^2 + 1)a\} S_{\kappa,} \\
a S_\kappa &= \frac{1}{1 + 2\kappa^2} S_\kappa 2\kappa^2 a \\
[a, S_\kappa] &= \frac{1}{1 + 2\kappa^2} S_{\kappa,z} - a \\
[a^\dagger, S_{\kappa,z}] &= \frac{1}{2\kappa^2} S_{\kappa,z} a^\dagger \\
[(a^\dagger - a),\, S_\kappa] &= S_\kappa \{ \frac{1}{2\kappa^2} a^\dagger + \frac{1}{1 + 2\kappa^2} a \}
\end{aligned} \tag{D.73}
$$

where

$$
S_\kappa = \frac{1}{2\pi\kappa^2} \int e^{-\frac{|z|^2}{2\kappa^2}} |z\rangle \langle z| \, d^2 z
$$

The two-parameter transformation we are considering is

$$
U(\theta^1, \theta^2) = \mathrm{e}^{-\mathrm{i}p_x\theta^1 - \mathrm{i}p_y\theta^2}. \tag{D.74}
$$

Since p_x and p_y commute,

$$
U(\theta^1, \theta^2) = \mathrm{e}^{-\mathrm{i}p_x\theta^1} \mathrm{e}^{-\mathrm{i}p_y\theta^2} = \mathrm{e}^{-\mathrm{i}p_y\theta^2} \mathrm{e}^{-\mathrm{i}p_x\theta^1} \tag{D.75}
$$

ρ_θ is

$$
\rho_\theta = U(\theta^1, \theta^2) \rho_0 U^\dagger(\theta^1, \theta^2)
$$

$$
\begin{aligned}
U_x(\theta^1) &= \mathrm{e}^{-\mathrm{i}p_x\theta^1} \\
U_y(\theta^2) &= \mathrm{e}^{-\mathrm{i}p_y\theta^2}
\end{aligned}
$$

We define $U_x(\theta^1)$ and $U_y(\theta^2)$ as follows.

$$U_x(\theta^1) = \mathrm{e}^{-\mathrm{i}p_x\theta^1}$$
$$U_y(\theta^2) = \mathrm{e}^{-\mathrm{i}p_y\theta^2}$$

Then, $U(\theta^1, \theta^2)$ is

$$U(\theta^1, \theta^2) = U_x(\theta^1)U_y(\theta^2)$$

With using $U_x(\theta^1)$ and $U_y(\theta^2)$, ρ_θ is

$$\begin{aligned}
\rho_\theta &= U_x(\theta^1)U_y(\theta^2)\rho_0(U_x(\theta^1)U_y(\theta^2))^\dagger \\
&= U_x(\theta^1)U_y(\theta^2)\rho_0 U_y^\dagger(\theta^2)U_x^\dagger(\theta^1) \\
&= U_x(\theta^1)U_y(\theta^2)\rho_{0\,a} \otimes \rho_{0\,b} U_y^\dagger(\theta^2)U_x^\dagger(\theta^1) \\
&= U_x(\theta^1)U_y(\theta^2)\rho_{0\,a} U_y^\dagger(\theta^2)U_x^\dagger(\theta^1) \otimes U_x(\theta^1)U_y(\theta^2)\rho_{0\,b} U_y^\dagger(\theta^2)U_x^\dagger(\theta^1)
\end{aligned}$$

We define $\rho^i_{\theta^i}$ by

$$\rho^1_{\theta^1} = \mathrm{e}^{-\mathrm{i}p_x\theta^1}\rho_0\mathrm{e}^{-\mathrm{i}p_x\theta^1} = U_x(\theta^1)\rho_0 U_x^\dagger(\theta^1) \tag{D.76}$$

As shown in Appendix D.1, the position x, y and the momentum p_x, p_y are

$$\begin{aligned}
x &= \frac{\lambda}{2}\{(a + a^\dagger) + (b + b^\dagger)\} \\
y &= \frac{\mathrm{i}\lambda}{2}\{(a - a^\dagger) - (b - b^\dagger)\} \\
p_x &= \frac{1}{\mathrm{i}2\lambda}\{(a - a^\dagger) + (b - b^\dagger)\} \\
p_y &= \frac{1}{2\lambda}\{(a + a^\dagger) - (b + b^\dagger)\}
\end{aligned}$$

D.6.1 RLD

We can calculate RLD in the same way given in Appendix C. By the definition, RLD is

$$\frac{\partial\rho_\theta}{\partial\theta^i} = \rho_\theta L_i^R \tag{D.77}$$

The left hand side of above for $i = 1$ is

$$\begin{aligned}
\frac{\partial\rho_\theta}{\partial\theta^1} &= -\mathrm{i}p_x U(\theta^1, \theta^2)\rho_0 U^\dagger(\theta^1, \theta^2)(\mathrm{i}p_x) \\
&= -\mathrm{i}U(\theta^1, \theta^2)[p_x, \rho_0]U^\dagger(\theta^1, \theta^2)
\end{aligned}$$

$$
\begin{aligned}
[p_x,\, \rho_0] &= \frac{1}{2\mathrm{i}\lambda}[(a-a^\dagger)\otimes I_b + I_a\otimes(a-a^\dagger),\, \rho_{0\,a}\otimes\rho_{0\,b}] \\
&= \frac{1}{2\mathrm{i}\lambda}[a-a^\dagger,\, \rho_{0\,a}]\otimes I_b + \frac{1}{2\mathrm{i}\lambda} I_a\otimes[(a-a^\dagger),\, \rho_{0\,b}]
\end{aligned}
$$

$\dfrac{\partial\rho_\theta}{\partial\theta^1}$ is

$$
\frac{\partial\rho_\theta}{\partial\theta^1} = -\frac{1}{2\lambda}[a-a^\dagger,\, \rho_{0\,a}]\otimes I_b - \frac{1}{2\lambda} I_a\otimes[(a-a^\dagger),\, \rho_{0\,b}]
$$

For $i=2$, we can calculate the same way.

$$
\frac{\partial\rho_\theta}{\partial\theta^2} = -\frac{\mathrm{i}}{2\lambda}[a+a^\dagger,\, \rho_{0\,a}]\otimes I_b + \frac{\mathrm{i}}{2\lambda} I_a\otimes[(a+a^\dagger),\, \rho_{0\,b}]
$$

RLD is

$$
\frac{\partial\rho_\theta}{\partial\theta^i} = \rho_\theta L^R_i
$$

Therefore, L^R_i is expressed as

$$
L^R_i = L^R_{a\,i}\otimes I_b + I_a\otimes L^R_{a\,i}
$$

With using the commutation relations given in Appendix B.3, $\dfrac{\partial\rho^1_{\theta^1}}{\partial\theta^1}$ is

$$
\frac{\partial\rho^1_{\theta^1}}{\partial\theta^1} = U_x(\theta^1)\rho_0\,\frac{1}{2\lambda}\,[\{\frac{1}{1+2\kappa_a^2}\,a + \frac{1}{2\kappa^2}\,a^\dagger\}\otimes I_b + I_a\otimes\{\frac{1}{1+2\kappa_b^2}\,a + \frac{1}{2\kappa_b^2}\,a^\dagger\}]\,U_x(\theta^1)^\dagger
$$

By the definition, RLD L^R_i is

$$
\frac{\partial\rho_\theta}{\partial\theta^i} = \rho_\theta L^R_i \tag{D.78}
$$

Therefore,

$$
\begin{aligned}
L^R_{01,\,a} &= \frac{1}{2\lambda}\,\{\frac{1}{1+2\kappa_a^2}\,a + \frac{1}{2\kappa_a^2}\,a^\dagger\} \\
L^R_{01,\,b} &= \frac{1}{2\lambda}\,\{\frac{1}{1+2\kappa_b^2}\,a + \frac{1}{2\kappa_b^2}\,a^\dagger\}
\end{aligned}
$$

$L^R_{01} = L^R_{01,\,a}\otimes I_b + I_a\otimes L^R_{01,\,b}$

$$
\frac{\partial\rho^1_{\theta^1}}{\partial\theta^1} = U_x(\theta^1)\rho_0\,(L^R_{01,\,a}\otimes I_b + I_a\otimes L^R_{01,\,b})U_x(\theta^1)^\dagger = U_x(\theta^1)\rho_0\,L^R_{01}U_x(\theta^1)^\dagger
$$

For $i=2$, ρ_{θ^2}, $[p_y, \rho_0]$ is

$$
(-\mathrm{i})[p_y, \rho_0] = -\frac{\mathrm{i}}{2\lambda}[(a^\dagger+a)\otimes I_b - I_a\otimes(a^\dagger+a),\, \rho_{0,\,a}\otimes\rho_{0,\,b}]
$$

With using the commutation relation in Appendix B.3 again, we obtain

$$(-\mathrm{i})[p_y,\rho_0] = -\frac{\mathrm{i}}{2\lambda}\rho_0\,[\{\frac{1}{1+2\kappa_a^2}\,(-a)+\frac{1}{2\kappa_a^2}\,a^\dagger\}\otimes I_b - I_a\otimes\{\frac{1}{1+2\kappa_b^2}\,(-a)-\frac{1}{2\kappa_b^2}\,(a^\dagger)\}]$$

Then, we have

$$L^R_{02} = L^R_{02,\,a}\otimes I_b + I_a\otimes L^R_{02,\,b} \tag{D.79}$$

where

$$\begin{aligned}
L^R_{02,\,a} &= -\frac{\mathrm{i}}{2\lambda}\,\{\frac{1}{1+2\kappa_a^2}\,(-a)+\frac{1}{2\kappa_a^2}\,(a^\dagger)\}\\
L^R_{02,\,b} &= \frac{\mathrm{i}}{2\lambda}\,\{\frac{1}{1+2\kappa_b^2}\,(-a)+\frac{1}{2\kappa_b^2}\,(a^\dagger)\}
\end{aligned}$$

$$\frac{\partial\rho^2_{\theta^2}}{\partial\theta^2} = U_y(\theta^2)\rho_0\,(L^R_{02,\,a}\otimes I_b + I_a\otimes L^R_{02,\,b})U_y(\theta^2)^\dagger = U_y(\theta^2)\rho_0\,L^R_{02}U_y(\theta^2)^\dagger$$

D.6.2 RLD Fisher information matrix

With the RLD, we move on to the calculation of Fisher information matrix, G^R.

$$[G^R]_{i\,j} = \mathrm{tr}\,[\rho_\theta L^R_{\theta\,j}L^{R\,\dagger}_{\theta\,i}]. \tag{D.80}$$

Since we consider the unitary transformation only, we have

$$\mathrm{tr}[\rho_\theta L^R_{\theta\,i}L^{R\,\dagger}_{\theta\,j}] = \mathrm{tr}[\rho_0 L^R_{0\,i}L^{R\,\dagger}_{0\,j}]$$

where

$$L^R_{\theta\,i} = U(\theta^1,(\theta^1)L^R_{0\,i}U^\dagger(\theta^1,(\theta^1)$$

By using $\rho_0 = \rho_{0,\,a}\otimes\rho_{0,\,b}$, we have

$$\begin{aligned}
\mathrm{tr}[\rho_0 L^R_{0i}L^{R\,\dagger}_{0j}] &= \mathrm{tr}[(\rho_{0,\,a}\otimes\rho_{0,\,b})(L^R_{0\,i,a}\otimes I_b + I_a\otimes L^R_{0\,j,b})(L^R_{0\,i,a}\otimes I_b + I_a\otimes L^R_{0\,j,b})^\dagger]\\
&= \mathrm{tr}[(\rho_{0,\,a}\otimes\rho_{0,\,b})\{(L^R_{0\,i,a})(L^R_{0\,j,a})^\dagger\otimes I_b + ((L^R_{0\,i,a})\otimes I_b)\,(I_a\otimes(L^R_{0\,i,b})^\dagger)\\
&+ I_a\otimes(L^R_{0\,i,b})(L^R_{0\,j,a})^\dagger\otimes I_b + I_a\otimes(L^R_{0\,i,b})(L^R_{0\,j,b})^\dagger\}]\\
&= \mathrm{tr}[\rho_{0,\,a}(L^R_{0\,i,a})(L^R_{0\,j,a})^\dagger]\,\mathrm{tr}[\rho_{0,\,b}] + \mathrm{tr}[\rho_{0,\,a}(L^R_{0\,i,a})]\,\mathrm{tr}[\rho_{0,\,b}]\;\;\mathrm{tr}[\rho_{0,\,a}]\mathrm{tr}[\rho_{0,\,b}(L^R_{0\,j,b})^\dagger]\\
&+ \mathrm{tr}[\rho_{0,\,a}]\,\mathrm{tr}[\rho_{0,\,b}(L^R_{0\,i,b})]\,\mathrm{tr}[\rho_{0,\,a}(L^R_{0\,j,a})^\dagger]\,\mathrm{tr}[\rho_{0,\,b}] + \mathrm{tr}[\rho_{0,\,a}]\,\mathrm{tr}[\rho_{0,\,b}(L^R_{0\,i,b})(L^R_{0\,j,b})^\dagger]
\end{aligned}$$

We can show $\mathrm{tr}\,[\rho_{0,\,a}(L^R_{0\,i,a})] = \mathrm{tr}\,[\frac{\partial\rho_{0,\,a}}{\partial\theta^i}] = \frac{\partial}{\partial\theta^i}\mathrm{tr}[\rho_{0,\,a}] = 0$, because $\mathrm{tr}[\rho_{0,\,a}] = 1$. Therefore, in the equation above, the second and the third terms vanish.

$$\mathrm{tr}\,[\rho_0 L^R_{0i}L^{R\,\dagger}_{0j}] = \mathrm{tr}[\rho_{0,\,a}(L^R_{0\,i,a})(L^R_{0\,j,a})^\dagger]\,\mathrm{tr}[\rho_{0,\,b}] + \mathrm{tr}[\rho_{0,\,a}]\,\mathrm{tr}[\rho_{0,\,b}(L^R_{0\,i,b})(L^R_{0\,j,b})^\dagger]$$

$$\mathrm{tr}\,[\rho_0 L^R_{0i} L^{R\dagger}_{0j}] = \mathrm{tr}[\rho_{0,a}(L^R_{0\,i,a})(L^R_{0\,j,a})^\dagger] + \mathrm{tr}[\rho_{0,b}(L^R_{0\,i,b})(L^R_{0\,j,b})^\dagger] \tag{D.81}$$

Let $g^R_{i\,j}$ be $[G^R]_{i\,j} = g^R_{i\,j}$. From the equation above, $g^R_{2\,2} = \mathrm{tr}[\rho_{0,a} L^R_{02} L^{R\dagger}_{02}]$ is

$$\mathrm{tr}[\rho_{0,a} L^R_{02} L^{R\dagger}_{02}] = \mathrm{tr}[\rho_{0,a} L^R_{02,a} L^{R\dagger}_{02,a}]] + \mathrm{tr}[\rho_{02,b} L^R_{02,b} L^{R\dagger}_{02,b}]$$

①tr $[\rho_{0,a} L^R_{02,a} L^{R\dagger}_{02,a}]$

$$\begin{aligned}
L^R_{02,a} &= -\frac{\mathrm{i}}{2\lambda}\{\frac{1}{1+2\kappa_a^2}(-a) + \frac{1}{2\kappa_b^2}(a^\dagger)\} \\
&= -\frac{\mathrm{i}}{2\lambda}(-\alpha a + \beta a^\dagger) \\
L^{R\dagger}_{02,a} &= \frac{\mathrm{i}}{2\lambda}(-\alpha a^\dagger + \beta a)
\end{aligned}$$

Let $\alpha = \dfrac{1}{1+2\kappa_a^2}$ and $\beta = \dfrac{1}{2\kappa_a^2}$.

$$\begin{aligned}
L^R_{02,a} L^{R\dagger}_{02,a} &= \frac{1}{4\lambda^2}(-\alpha a + \beta a^\dagger)(-\alpha a^\dagger + \beta a) \\
&= \frac{1}{4\lambda^2}(\alpha^2 a a^\dagger - \alpha\beta a^2 - \alpha\beta a^{\dagger\,2} + \beta^2 a^\dagger a) \\
\mathrm{tr}[\rho_{0,a} L^{2\,R}_{0,a} L^{2\,R\dagger}_{0,a}] &= \frac{1}{4\lambda^2}\,\mathrm{tr}\,[\rho_{0,a}\{\alpha^2(a^\dagger a + 1) + \beta^2 a^\dagger a\}] \\
&= \frac{1}{4\lambda^2}\{\alpha^2(1 + \frac{1}{\beta}) + \beta^2\frac{1}{\beta}\} \\
&= \frac{1}{4\lambda^2}\{\frac{1}{(1+2\kappa_a^2)^2}(1+2\kappa_a^2) + \frac{1}{2\kappa_a^2}\} \\
&= \frac{1}{4\lambda^2}(\frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2})
\end{aligned} \tag{D.82}$$

②tr $[\rho_{0,b} L^{2\,R}_{0,b} L^{2\,R\dagger}_{0,b}]$

$$\begin{aligned}
L^{2\,R}_{0,b} &= \frac{\mathrm{i}}{2\lambda}(-\alpha a + \beta a^\dagger) \\
L^{2\,R\dagger}_{0,b} &= -\frac{\mathrm{i}}{2\lambda}(-\alpha a^\dagger + \beta a)
\end{aligned}$$

Since $L^{2\,R}_{0,b} L^{2\,R\dagger}_{0,b} = L^{2\,R}_{0,a} L^{2\,R\dagger}_{0,a}$, the result is the same.

$$\mathrm{tr}[\rho_{0,b} L^{1\,R}_{0,b} L^{1\,R\dagger}_{0,b}] = \frac{1}{4\lambda^2}(\frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2}).$$

By adding both, we have

$$\mathrm{tr}[\rho_{0,a} L^R_{02} L^{R\dagger}_{02}] = \mathrm{tr}[\rho_{0,a} L^{2\,R}_{0,a} L^{2\,R\dagger}_{0,a}] + \mathrm{tr}[\rho_{0,b} L^{2\,R}_{0,b} L^{2\,R\dagger}_{0,b}] = \frac{1}{2\lambda}^2(\frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2})$$

$g_{1\,1}^{R} = \mathrm{tr}[, [\rho_{0,a} L_0^{1\,R} L_0^{1\,R\dagger}]$

$$L_0^{1\,R} = L_{0,a}^{1\,R} \otimes I_b + I_a \otimes L_{0,a}^{1\,R}$$

①$\mathrm{tr}[\rho_{0,a} L_{0,a}^{1\,R} L_{0,a}^{1\,R\dagger}]$

$$\begin{aligned}
L_{0,a}^{1\,R} &= \frac{1}{2\lambda}\,(\alpha a + \beta a^\dagger) \\
L_{0,a}^{1\,R\dagger} &= \frac{1}{2\lambda}\,(\alpha a^\dagger + \beta a) \\
\alpha &= \frac{1}{1 + 2\kappa_a^2} \\
\beta &= \frac{1}{2\kappa_a^2} \\
\frac{1}{\frac{1}{2\lambda}^2} L_{0,a}^{1\,R} L_{0,a}^{1\,R\dagger} &= (\alpha a + \beta a^\dagger)(\alpha a^\dagger + \beta a) \\
&= \alpha^2 a a^\dagger + \alpha\beta a^2 + \alpha\beta a^{\dagger\,2} + \beta^2 a^\dagger a \\
\mathrm{tr}[\rho_{0,a} L_{0,a}^{1\,R} L_{0,a}^{1\,R\dagger}] &= \frac{1}{4\lambda^2}\,\mathrm{tr}\,[\alpha^2(a^\dagger a + 1) + \beta^2(a^\dagger a)] \\
&= \frac{1}{4\lambda^2}\,\{\alpha^2(1 + \frac{1}{\beta}) + \beta^2 \frac{1}{\beta}\} \\
&= \frac{1}{4\lambda^2}\,\{\frac{1}{(1 + 2\kappa_a^2)^2}(1 + 2\kappa_a^2) + \frac{1}{2\kappa_a^2}\} \\
&= \frac{1}{4\lambda^2}\,(\frac{1}{1 + 2\kappa_a^2} + \frac{1}{2\kappa_a^2}) \qquad \text{(D.83)}
\end{aligned}$$

②$\mathrm{tr}[\rho_{0,b} L_{0,b}^{1\,R} L_{0,b}^{1\,R\dagger}]$
From $L_{0,b}^{1\,R} L_{0,b}^{1\,R\dagger} = L_{0,a}^{2\,R} L_{0,a}^{2\,R\dagger}$,

$$\mathrm{tr}[\rho_{0,b} L_{0,b}^{1\,R} L_{0,b}^{1\,R\dagger}] = \frac{1}{4\lambda^2}\,(\frac{1}{1 + 2\kappa_b^2} + \frac{1}{2\kappa_b^2})$$

By adding up all, we have

$$\begin{aligned}
\mathrm{tr}[\rho_{0,a} L_{02}^{R} L_{02}^{R\dagger}] &= \mathrm{tr}[\rho_{0,a} L_{0,a}^{2\,R} L_{0,a}^{2\,R\dagger}] + \mathrm{tr}[\rho_{0,b} L_{0,b}^{2\,R} L_{0,b}^{2\,R\dagger}] \\
&= \frac{1}{4\lambda^2}\,(\frac{1}{1 + 2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1 + 2\kappa_b^2} + \frac{1}{2\kappa_b^2})
\end{aligned}$$

$g_{1\,2}^{R} = \mathrm{tr}[\rho_{0,a} L_{02}^{R} L_0^{1\,R\dagger}]$

①$\mathrm{tr}[\rho_{0,a} L^{2R}_{0,a} L^{1R\dagger}_{0,a}]$

$$
\begin{aligned}
L^{2R}_{0,a} &= -\frac{\mathrm{i}}{2\lambda}(-\alpha a + \beta a^\dagger) \\
L^{1R\dagger}_{0,a} &= \frac{1}{2\lambda}(\alpha a^\dagger + \beta a) \\
L^{2R}_{0,a} L^{1R\dagger}_{0,a} &= -\frac{\mathrm{i}}{4\lambda^2}(-\alpha a + \beta a^\dagger)(\alpha a^\dagger + \beta a) \\
&= \frac{\mathrm{i}}{4\lambda^2}(\alpha a - \beta a^\dagger)(\alpha a^\dagger + \beta a) \\
&= \frac{\mathrm{i}}{4\lambda^2}(\alpha^2 a a^\dagger + \alpha\beta a^2 - \beta\alpha (a^\dagger)^2 - \beta^2 a^\dagger a) \\
&= \frac{\mathrm{i}}{4\lambda^2}\{\alpha^2 (a a^\dagger + 1) + \alpha\beta - \beta^2 (a^\dagger)^2 - \beta^2 a^\dagger a\} \\
\mathrm{tr}[,[\rho_{0,a} L^{2R}_{0,a} L^{1R\dagger}_{0,a}] &= \frac{\mathrm{i}}{4\lambda^2}\,\mathrm{tr}\,[\rho_{0,a}\{\alpha^2 (a a^\dagger + 1) + \alpha\beta - \beta^2 (a^\dagger)^2 - \beta^2 a^\dagger a\}] \\
&= \frac{\mathrm{i}}{4\lambda^2}\left(\frac{1}{1+2\kappa_a^2} - \frac{1}{2\kappa_a^2}\right) \\
&= -\frac{\mathrm{i}}{4\lambda^2}\frac{1}{2\kappa_a^2(1+2\kappa_a^2)}
\end{aligned}
$$

②$\mathrm{tr}[\rho_{0,b} L^{2R}_{0,b} L^{1R\dagger}_{0,b}]$

$$
\begin{aligned}
L^{2R}_{0,b} &= \frac{\mathrm{i}}{2\lambda}(-\alpha a + \beta a^\dagger) \\
L^{1R\dagger}_{0,b} &= \frac{1}{2\lambda}(\alpha a^\dagger + \beta a)
\end{aligned}
$$

From $L^{2R}_{0,b} L^{1R\dagger}_{0,b} = -L^{2R}_{0,a} L^{1R\dagger}_{0,a}$,

$$
\mathrm{tr}\,[\rho_{0,b} L^{2R}_{0,b} L^{1R\dagger}_{0,b}] = -\frac{\mathrm{i}}{4\lambda^2}\,\mathrm{tr}\,[\rho_{0,b}(\alpha^2 a a^\dagger + \alpha\beta a^2 - \beta\alpha (a^\dagger)^2 - \beta^2 a^\dagger a)] = \frac{\mathrm{i}}{4\lambda^2}\frac{1}{2\kappa_b^2(1+2\kappa_b^2)}
$$

By adding them up, we have

$$
\begin{aligned}
\mathrm{tr}[\rho_{0,a} L^{R}_{02} L^{1R\dagger}_{0}] &= \mathrm{tr}[\rho_{0,a} L^{2R}_{0,a} L^{1R\dagger}_{0,a}] + \mathrm{tr}[\rho_{0,b} L^{2R}_{0,b} L^{1R\dagger}_{0,b}] \\
&= -\frac{\mathrm{i}}{4\lambda^2}\left\{\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}\right\}
\end{aligned}
$$

Finally, we obtain the RLD Fisher information matrix, $G^R(\theta)$.

$$
G^R(\theta) = \frac{1}{4\lambda^2}\begin{pmatrix} \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} & -\mathrm{i}\left\{\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}\right\} \\ \mathrm{i}\left\{\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}\right\} & \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} \end{pmatrix}
$$

D.6.3 RLD bound

Let $G^R(\theta)$ be

$$
G^R(\theta) = \frac{1}{4\lambda^2}\begin{pmatrix} g_1 & -\mathrm{i}\,g_2 \\ \mathrm{i}\,g_2 & g_1 \end{pmatrix}. \tag{D.84}
$$

Then, its inverse $(G^R(\theta))^{-1}$ is

$$
\begin{aligned}
(G^R(\theta))^{-1} &= \frac{4\lambda^2}{\{(g_1)^2 - (g_2)^2)\}} \begin{pmatrix} g_1 & \mathrm{i}\, g_2 \\ -\mathrm{i}\, g_2 & g_1 \end{pmatrix} \\
&= \frac{4\lambda^2}{(g_1 + g_2)(g_1 - g_2)} \begin{pmatrix} g_1 & \mathrm{i}\, g_2 \\ -\mathrm{i}\, g_2 & g_1 \end{pmatrix}
\end{aligned}
\tag{D.85}
$$

$$
\begin{aligned}
g_1 + g_2 &= \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} + \frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)} \\
&= \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} + \frac{1}{2\kappa_a^2} - \frac{1}{1+2\kappa_a^2} - \frac{1}{2\kappa_b^2} + \frac{1}{1+2\kappa_b^2} \\
&= 2(\frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2}) \\
\frac{1}{g_1 + g_2} &= \frac{1}{2}\frac{2\kappa_a^2(1+2\kappa_b^2)}{1+2\kappa_a^2+2\kappa_b^2} \\
g_1 - g_2 &= \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} - \{\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}\} \\
&= \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} - \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_b^2} - \frac{1}{1+2\kappa_b^2} \\
&= 2(\frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_b^2}) \\
\frac{1}{g_1 - g_2} &= \frac{1}{2}\frac{2\kappa_b^2(1+2\kappa_a^2)}{1+2\kappa_a^2+2\kappa_b^2} \\
(G^R(\theta))^{-1} &= \frac{4\lambda^2}{4}\frac{2\kappa_a^2 2\kappa_b^2(1+2\kappa_a^2)(1+2\kappa_b^2)}{(1+2\kappa_a^2+2\kappa_b^2)^2} \begin{pmatrix} g_1 & -\mathrm{i}\, g_2 \\ \mathrm{i}\, g_2 & g_1 \end{pmatrix} = \lambda^2 \frac{2\kappa_a^2 2\kappa_b^2(1+2\kappa_a^2)(1+2\kappa_b^2)}{(1+2\kappa_a^2+2\kappa_b^2)^2} \begin{pmatrix} g_1 & -\mathrm{i}\, g_2 \\ \mathrm{i}\, g_2 & g_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
g^{R,11} &= \lambda^2(\frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2})(\frac{1}{2\kappa_b^2} + \frac{1}{1+2\kappa_a^2})(\frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2}) \\
&= \lambda^2(\frac{1}{\frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2}} + \frac{1}{\frac{1}{2\kappa_b^2} + \frac{1}{1+2\kappa_a^2}}) \\
&= \lambda^2\{\frac{2\kappa_a^2(1+2\kappa_b^2)}{1+2\kappa_a^2+2\kappa_b^2} + \frac{2\kappa_b^2(1+2\kappa_a^2)}{1+2\kappa_a^2+2\kappa_b^2}\} \\
&= \lambda^2\frac{2\kappa_a^2 + 2\kappa_b^2 + 8\kappa_a^2\kappa_b^2}{1+2\kappa_a^2+2\kappa_b^2}
\end{aligned}
\tag{D.86}
$$

$$
\begin{aligned}
g^{R,12} &= \lambda^2 \frac{\mathrm{i}}{(\frac{1}{2\kappa_a^2}+\frac{1}{1+2\kappa_b^2})(\frac{1}{2\kappa_b^2}+\frac{1}{1+2\kappa_a^2})}\{\frac{1}{2\kappa_a^2}-\frac{1}{1+2\kappa_a^2}-(\frac{1}{2\kappa_b^2}-\frac{1}{1+2\kappa_b^2})\} \\
&= \lambda^2 \frac{\mathrm{i}}{(\frac{1}{2\kappa_a^2}+\frac{1}{1+2\kappa_b^2})(\frac{1}{2\kappa_b^2}+\frac{1}{1+2\kappa_a^2})}\{\frac{1}{2\kappa_a^2}+\frac{1}{1+2\kappa_b^2}-(\frac{1}{2\kappa_b^2}+\frac{1}{1+2\kappa_a^2})\} \\
&= \mathrm{i}\lambda^2(\frac{1}{\frac{1}{2\kappa_b^2}+\frac{1}{1+2\kappa_a^2}}-\frac{1}{\frac{1}{2\kappa_a^2}+\frac{1}{1+2\kappa_b^2}}) \\
&= \mathrm{i}\lambda^2\{\frac{2\kappa_b^2(1+2\kappa_a^2)}{1+2\kappa_a^2+2\kappa_b^2}-\frac{2\kappa_a^2(1+2\kappa_b^2)}{1+2\kappa_a^2+2\kappa_b^2}\} \\
&= -\mathrm{i}\lambda^2\frac{2\kappa_a^2-2\kappa_b^2}{1+2\kappa_a^2+2\kappa_b^2}
\end{aligned}
$$

$$
(G^R(\theta))^{-1} = \frac{\lambda^2}{1+2\kappa_a^2+2\kappa_b^2}\begin{pmatrix} 2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 & \mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) \\ -\mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) & 2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 \end{pmatrix} \tag{D.87}
$$

D.6.4 Quantum Cramer-Rao inequality

Quantum Cramer-Rao inequality is

$$
V_\theta \geq G_\theta^{-1} \tag{D.88}
$$

On the lefthand side, multiply V_θ by $(1, \mathrm{i}\eta)$ from the left and by $(1, -\mathrm{i}\eta)^T$ from the right.

$$
\begin{aligned}
\begin{pmatrix} 1 & \mathrm{i}\eta \end{pmatrix}\begin{pmatrix} V_{\theta,11} & V_{\theta,12} \\ V_{\theta,21} & V_{\theta,22} \end{pmatrix}\begin{pmatrix} 1 \\ -\mathrm{i}\eta \end{pmatrix} &= \begin{pmatrix} 1 & \mathrm{i}\eta \end{pmatrix}\begin{pmatrix} V_{\theta,11}-\mathrm{i}\eta V_{\theta,12} \\ V_{\theta,21}-\mathrm{i}\eta V_{\theta,22} \end{pmatrix} \\
&= V_{\theta,11}-\mathrm{i}\eta V_{\theta,12}+\mathrm{i}\eta(V_{\theta,21}-\mathrm{i}\eta V_{\theta,22}) \\
&= V_{\theta,11}+\eta^2 V_{\theta,22}
\end{aligned}
$$

$$
\therefore \begin{pmatrix} 1 & \mathrm{i}\eta \end{pmatrix}\begin{pmatrix} V_{\theta,11} & V_{\theta,12} \\ V_{\theta,21} & V_{\theta,22} \end{pmatrix}\begin{pmatrix} 1 \\ -\mathrm{i}\eta \end{pmatrix} = V_{\theta,11}+\eta^2 V_{\theta,22}
$$

We do the same to $(G_\theta^R)^{-1}$. Let $(G_\theta^R)^{-1}$ be

$$
\begin{pmatrix} G^{-1}_{\theta,11} & G^{-1}_{\theta,12} \\ G^{-1}_{\theta,21} & G^{-1}_{\theta,22} \end{pmatrix} = \begin{pmatrix} A & \mathrm{i}B \\ -\mathrm{i}B & A \end{pmatrix}.
$$

$$
\begin{pmatrix} 1 & \mathrm{i}\eta \end{pmatrix}\begin{pmatrix} A & \mathrm{i}B \\ -\mathrm{i}B & A \end{pmatrix}\begin{pmatrix} 1 \\ -\mathrm{i}\eta \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i}\eta \end{pmatrix}\begin{pmatrix} A+\eta B \\ -\mathrm{i}B-\mathrm{i}\eta A \end{pmatrix} = A+\eta B+\mathrm{i}\eta(-\mathrm{i}B-\mathrm{i}\eta A) = (1+\eta^2)A+2\eta B
$$

From $V_\theta \geq (G_\theta^R)^{-1}$, we obtain the inequality below.

$$
V_{\theta,11}+\eta^2 V_{\theta,22} \geq (1+\eta^2)A+2\eta B
$$

$$\therefore (V_{\theta,22} - A)\eta^2 - 2B\eta + V_{\theta,11} - A \geq 0 \tag{D.89}$$

For the lefthand side to have the minimum, the second derivative of the lefthand side in respect to η must be positive.

$$2(V_{\theta,22} - A) > 0$$
$$V_{\theta,22} - A > 0$$

The minimum is given at the first derivative in respect to η being zero. Let the first derivative in respect to η be at η_0. Then, η_0 is

$$\eta_0 = \frac{2B}{2(V_{\theta,22} - A)} = -\frac{B}{V_{\theta,22} - A}$$
$$V_{\theta,22} - A = -\frac{B}{\eta_0}$$

$$\frac{B}{\eta_0}\eta_0^2 - 2B\eta_0 + V_{\theta,11} - A \geq 0$$
$$-B\eta_0 + V_{\theta,11} - A \geq 0$$
$$-\frac{B^2}{V_{\theta,22} - A} + V_{\theta,11} - A \geq 0 \tag{D.90}$$

Multiply $V_{\theta,22} - A$ on the both sides. No change in the inequality, because $V_{\theta,22} - A \geq 0$

$$-B^2 + (V_{\theta,11} - A)(V_{\theta,22} - A) \geq 0$$

Quantum Cramer-Rao inequality is

$$(V_{\theta,11})(V_{\theta,22}) - A(V_{\theta,11} + V_{\theta,22}) + A^2 \geq B^2 \tag{D.91}$$

D.6.5 SLD Fisher information matrix

As given in the RLD section above, $\frac{\partial\rho_\theta}{\partial\theta^1}$ is

$$\frac{\partial\rho_\theta}{\partial\theta^1} = -\frac{1}{2\lambda}[a - a^\dagger, \rho_{0a}] \otimes I_b - \frac{1}{2\lambda}I_a \otimes [(a - a^\dagger), \rho_{0b}]$$

$\frac{\partial\rho_\theta}{\partial\theta^2}$ is

$$\frac{\partial\rho_\theta}{\partial\theta^2} = -\frac{\mathrm{i}}{2\lambda}[a + a^\dagger, \rho_{0a}] \otimes I_b + \frac{\mathrm{i}}{2\lambda}I_a \otimes [(a + a^\dagger), \rho_{0b}]$$

If we compare these with the SLD of one-parameter in Appendix D.5 about Model 1, we find that in the part "a", $\frac{\partial \rho_{0,a}}{\partial \theta^i}$ is the same, except for the cofficients. ($\frac{1}{\lambda}$ for Model 1 and $\frac{1}{2\lambda}$ for Model 2.) Then, we obtain

$$\begin{aligned}
L^S_{0\,i} &= L^S_{0\,i,a} \otimes I_b + I_a \otimes L^S_{0\,1,b} \\
L^S_{0\,1,a} &= \frac{1}{\lambda}\frac{1}{1+4\kappa_a^2}(a + a^\dagger) \\
L^S_{0\,1,b} &= \frac{1}{\lambda}\frac{1}{1+4\kappa_b^2}(a + a^\dagger) \\
L^S_{0\,2,a} &= \frac{\mathrm{i}}{\lambda}\frac{1}{1+4\kappa_b^2}(a - a^\dagger) \\
L^S_{0\,2,b} &= -\frac{\mathrm{i}}{\lambda}\frac{1}{1+4\kappa_b^2}(a - a^\dagger)
\end{aligned}$$

$\mathrm{tr}\,[\rho_{0\,a} L^S_{0\,1,a} L^S_{0\,1,a}]$ is

$$\begin{aligned}
\mathrm{tr}\,[\rho_{0\,a} L^S_{0\,1,a} L^S_{0\,1,a}] &= \mathrm{tr}\,[\rho_{0\,a}\frac{1}{\lambda^2}\frac{1}{(1+4\kappa_a^2)^2}(a^\dagger + a)^2] \\
&= \frac{1}{\lambda^2}\frac{1}{(1+4\kappa_a^2)^2}\,\mathrm{tr}\,[\rho_{0\,a}\{(a^\dagger)^2 + a^\dagger a + a a^\dagger + a^2\}] \\
&= \frac{1}{\lambda^2}\frac{1}{(1+4\kappa_a^2)^2}\,\mathrm{tr}\,[\rho_{0\,a}(2a^\dagger a + 1)] \\
&= \frac{1}{\lambda^2}\frac{1}{(1+4\kappa_a^2)^2}(1+4\kappa_a^2) \\
&= \frac{1}{\lambda^2}\frac{1}{1+4\kappa_a^2}
\end{aligned}$$

$\mathrm{tr}\,[\rho_{0\,b} L^S_{0\,1,a} L^S_{0\,1,b}]$ can be calculated in the same way.

$$tr\,[\rho_{0\,b} L^S_{0\,1,b} L^S_{0\,1,b}] = \frac{1}{\lambda^2}\frac{1}{1+4\kappa_b^2}$$

$$\mathrm{tr}\,[\rho_0 L^S_{0\,1} L^S_{0\,1}] = \frac{1}{\lambda^2}\left(\frac{1}{1+4\kappa_a^2} + \frac{1}{1+4\kappa_b^2}\right)$$

$\mathrm{tr}\,[\rho_{0\,a} L^S_{0\,2,a} L^S_{0\,1,a}]$ is

$$\begin{aligned}
\mathrm{tr}\,[\rho_{0\,a} L^S_{0\,2,a} L^S_{0\,1,a}] &= \mathrm{tr}\,[\rho_{0\,a}\frac{\mathrm{i}}{\lambda^2}\frac{1}{(1+4\kappa_a^2)^2}(a - a^\dagger)(a + a^\dagger)] \\
&= \frac{1}{\lambda^2}\frac{\mathrm{i}}{(1+4\kappa_a^2)^2}\,\mathrm{tr}\,[\rho_{0\,a}\{a^2 + a a^\dagger - a^\dagger a - (a^\dagger)^2\}] \\
&= \frac{1}{\lambda^2}\frac{\mathrm{i}}{(1+4\kappa_a^2)^2}\,\mathrm{tr}\,[\rho_{0\,a}\{a^2 + 1 - (a^\dagger)^2\}] \\
&= \frac{1}{\lambda^2}\frac{\mathrm{i}}{(1+4\kappa_a^2)^2}
\end{aligned}$$

$\mathrm{tr}\,[\rho_{0\,b}L^S_{0\,1,b}L^S_{0\,1,b}]$ can be calculated in the same way.

$$\mathrm{tr}\,[\rho_{0\,b}L^S_{0\,1,b}L^S_{0\,1,b}] = -\frac{1}{\lambda^2}\frac{\mathrm{i}}{(1+4\kappa_b)^2}$$

Therefore, we have

$$\mathrm{tr}\,[\rho_0 L^S_{0\,2}L^S_{0\,1}] = \frac{\mathrm{i}}{\lambda^2}\{\frac{1}{(1+4\kappa_a)^2} - \frac{1}{(1+4\kappa_b)^2}\}$$

Finally, we have

$$g^S_{1\,1} = \mathrm{Re}\,\mathrm{tr}\,[\rho_{0\,b}L^S_{0\,1,b}L^S_{0\,1,b}] = -\frac{1}{\lambda^2}\frac{\mathrm{i}}{(1+4\kappa_b)^2},$$

and

$$g^S_{1\,2} = \mathrm{Re}\,\mathrm{tr}\,[\rho_0 L^S_{0\,2}L^S_{0\,1}] = 0.$$

$[G^S_\theta]_{i\,j} = g^S_{i\,j}$ is

$$G^S_\theta = \frac{1}{\lambda^2}(\frac{1}{1+4\kappa_a^2} + \frac{1}{1+4\kappa_b^2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\lambda^2}\frac{2+4\kappa_a^2+4\kappa_b^2}{(1+4\kappa_a^2)(1+4\kappa_b^2)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Its inverse is

$$(G^S_\theta)^{-1} = \frac{\lambda^2(1+4\kappa_a^2)(1+4\kappa_b^2)}{2+4\kappa_a^2+4\kappa_b^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

D.6.6 Z matrix

SLD Fisher information matrix G^S_θ and $\tilde{G^S}_\theta$ are

$$[G^S_\theta]_{i\,j} = \mathrm{Re}\,\mathrm{tr}\,[\rho_0 L^S_{0\,j}L^S_{0\,i}]$$

$$[\tilde{G^S}_\theta]_{i\,j} = \mathrm{tr}\,[\rho_0 L^S_{0\,j}L^S_{0\,i}]$$

Therefore, $\tilde{G^S_\theta}$ is

$$\begin{aligned}
\tilde{G^S_\theta} &= \frac{1}{\lambda^2}\begin{pmatrix}\frac{1}{1+4\kappa_a^2}+\frac{1}{1+4\kappa_b^2} & \mathrm{i}(\frac{1}{(1+4\kappa_a)^2}-\frac{1}{(1+4\kappa_b)^2})\\ -\mathrm{i}(\frac{1}{(1+4\kappa_a)^2}-\frac{1}{(1+4\kappa_b)^2}) & \frac{1}{1+4\kappa_a^2}+\frac{1}{1+4\kappa_b^2}\end{pmatrix}\\
&= \frac{1}{\lambda^2}\begin{pmatrix}\frac{2+8\kappa_a^2+8\kappa_b^2}{(1+4\kappa_a^2)(1+4\kappa_b^2)} & \mathrm{i}\,\frac{(1+4\kappa_b^2)^2-(1+4\kappa_a^2)^2}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2}\\ -\mathrm{i}\,\frac{(1+4\kappa_b^2)^2-(1+4\kappa_a^2)^2}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2} & \frac{2+8\kappa_a^2+8\kappa_b^2}{(1+4\kappa_a^2)(1+4\kappa_b^2)}\end{pmatrix}\\
&= \frac{1}{\lambda^2}\begin{pmatrix}\frac{2+4\kappa_a^2+4\kappa_b^2}{(1+4\kappa_a^2)(1+4\kappa_b^2)} & \mathrm{i}\,\frac{(4\kappa_b^2-4\kappa_a^2)(2+4\kappa_a^2+4\kappa_a^2)}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2}\\ -\mathrm{i}\,\frac{(4\kappa_b^2-4\kappa_a^2)(2+4\kappa_a^2+4\kappa_a^2)}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2} & \frac{2+4\kappa_a^2+4\kappa_b^2}{(1+4\kappa_a^2)(1+4\kappa_b^2)}\end{pmatrix}\\
&= \frac{1}{\lambda^2}\frac{2+4\kappa_a^2+4\kappa_b^2}{(1+4\kappa_a^2)^2(1+4\kappa_b^2)^2}\begin{pmatrix}(1+4\kappa_a^2)(1+4\kappa_b^2) & \mathrm{i}\,(4\kappa_b^2-4\kappa_a^2)\\ -\mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) & (1+4\kappa_a^2)(1+4\kappa_b^2)\end{pmatrix}
\end{aligned}$$

We can calculate Z matrix by the formula below.

$$Z(\theta) = (G^S_\theta)^{-1}\tilde{G^S}_\theta(G^S\theta)^{-1}$$

Then, $Z(\theta)$ is

$$\begin{aligned}
Z(\theta) &= \frac{\lambda^2}{2+4\kappa_a^2+4\kappa_b^2}\begin{pmatrix} (1+4\kappa_a^2)(1+4\kappa_b^2) & \mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) \\ -\mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) & (1+4\kappa_a^2)(1+4\kappa_b^2) \end{pmatrix} \\
&= \frac{\lambda^2}{2+4\kappa_a^2+4\kappa_b^2}\begin{pmatrix} 1+4\kappa_a^2+4\kappa_b^2+16\kappa_a^2\kappa_b^2 & \mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) \\ -\mathrm{i}\,(4\kappa_b^2-4\kappa_a^2) & 1+4\kappa_a^2+4\kappa_b^2+16\kappa_a^2\kappa_b^2 \end{pmatrix} \\
&= \frac{\lambda^2}{1+2\kappa_a^2+2\kappa_b^2}\begin{pmatrix} \frac{1}{2}+2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 & \mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) \\ -\mathrm{i}\,(2\kappa_b^2-2\kappa_a^2) & \frac{1}{2}+2\kappa_a^2+2\kappa_b^2+8\kappa_a^2\kappa_b^2 \end{pmatrix}
\end{aligned}$$

From (D.87), we find $Z(\theta) \neq (G^R_\theta)^{-1}$. Model 2 is not D-invariant.

D.6.7 The variance of x and y : Model 2

$$x = \frac{\lambda}{2}\{(a+a^\dagger)\otimes I_b + I_a\otimes(a+a^\dagger)\}$$

$$\mathrm{tr}\,[\rho_0 x] = \frac{\lambda}{2}\mathrm{tr}\,[\rho_{0\,a}(a+a^\dagger)] + \frac{\lambda}{2}\mathrm{tr}\,[\rho_{0\,b}(a+a^\dagger)] = 0 \tag{D.92}$$

$$\begin{aligned}
\mathrm{tr}\,[\rho_0 x^2] &= \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,a}(a+a^\dagger)^2] + \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,b}(a+a^\dagger)^2] \\
&= \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,a}\{a^2+aa^\dagger+a^\dagger a+(a^\dagger)^2\}] + \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,b}\{a^2+aa^\dagger+a^\dagger a+(a^\dagger)^2\}] \\
&= \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,a}\{a^2+2a^\dagger a+1+(a^\dagger)^2\}] + \frac{\lambda^2}{4}\mathrm{tr}\,[\rho_{0\,a}\{a^2+2a^\dagger a+1+(a^\dagger)^2\}] \\
&= \frac{\lambda^2}{4}(1+4\kappa_a^2) + \frac{\lambda^2}{4}(1+4\kappa_b^2) \\
&= \frac{\lambda^2}{2}(1+2\kappa_a^2+2\kappa_b^2)
\end{aligned}$$

$$\therefore\ (\Delta x)^2 = \langle x^2\rangle_0 - \langle x\rangle_0^2 = \frac{\lambda^2}{4}(1+4\kappa_a^2) + \frac{\lambda^2}{4}(1+4\kappa_b^2) = \frac{\lambda^2}{2}(1+2\kappa_a^2+2\kappa_b^2)$$

We can determine $(\Delta y)^2$ in the same way. The result is

$$(\Delta y)^2 = \frac{\lambda^2}{2}(1+2\kappa_a^2+2\kappa_b^2)$$

$$(\Delta x)^2(\Delta y)^2 = \frac{\lambda^4}{4}(1+2\kappa_a^2+2\kappa_b^2)^2$$

$$(\Delta x)(\Delta y) = \frac{\lambda^2}{2}(1+2\kappa_a^2+2\kappa_b^2)$$

The Heisenberg-Robertson uncertainty relation gives $(\Delta x)(\Delta y) \geq \frac{1}{2}|\langle[x, y]\rangle|$. If we use it, $(\Delta x)(\Delta y)$ is

$$
\begin{aligned}
[x, y] &= -\frac{\mathrm{i}}{4\lambda^2}[a^\dagger + a + b^\dagger + b, a^\dagger - a - (b^\dagger - b)] \\
&= -\frac{\mathrm{i}}{4\lambda^2}([a^\dagger + a, a^\dagger - a] - [b^\dagger + b, b^\dagger - b]) \\
\therefore [x, y] &= 0
\end{aligned} \tag{D.93}
$$

$(\Delta x)(\Delta y)$ based on the Heisenberg-Robertson uncertainty relation is $(\Delta x)(\Delta y) \geq 0$

Appendix E

Reference state 0 and 1

We calculate $(G^R)^{-1}$, $(G^S)^{-1}$, Z for the reference state 0 and 1, below. We used the formulae for the pure state given in [9].

Definition of the four reference states

Reference state 0 : $\rho_0^{(0)} = |0\rangle_a \langle 0| \otimes |0\rangle_b \langle 0|$

Reference state 1 : $\rho_0^{(1)} = \sum f(n) |n\rangle_a \langle n| \otimes |0\rangle_b \langle 0|$

Reference state 2 : $\rho_0^{(2)} = \rho_{0,a} \otimes \rho_{0\,b}$ (Model 1 : $\kappa \to 0$, Model 2 : $\kappa_b \to 0$ after Fisher information matrices are evaluated.)

Reference state 3 : $\rho_0^{(3)} = \rho_{0,a} \otimes \rho_{0\,b}$

E.1 Reference state 0

E.1.1 Model 1

$\rho_0^{(0)} = |0\rangle_a \langle 0| \otimes |0\rangle_b \langle 0|$

The unitary transformation of Model1 is

$$\begin{aligned}
U(\theta) &= \mathrm{e}^{-\mathrm{i}\pi_x\theta^1 - \mathrm{i}\pi_y\theta^1} \\
&= \mathrm{e}^{-\frac{\mathrm{i}\theta^1}{\lambda}\mathrm{i}(a^\dagger - a) - \frac{\mathrm{i}\theta^2}{\lambda}(a^\dagger + a)} \\
&= \mathrm{e}^{\frac{\theta^1}{\lambda}(a^\dagger - a) - \frac{\mathrm{i}\theta^2}{\lambda}(a^\dagger + a)} \\
&= \mathrm{e}^{\frac{a^\dagger}{\lambda}(\theta^1 - \mathrm{i}\theta^2) - \frac{a}{\lambda}(\theta^1 + \mathrm{i}\theta^2)} \\
&= \mathrm{e}^{za^\dagger - z^* a} \\
&= \mathrm{e}^{\frac{-|z|^2}{2}}\mathrm{e}^{za^\dagger}e^{-z^* a} = \mathrm{e}^{\frac{|z|^2}{2}}e^{-z^* a}\mathrm{e}^{za^\dagger}
\end{aligned} \tag{E.1}$$

where $z = \frac{1}{\lambda}(\theta^1 - \mathrm{i}\theta^2)$. From (E.1), $\partial_z U(\theta)$ and $\partial_{z^*} U(\theta)$ are

$$\begin{aligned}
\partial_z U(\theta) &= (-\frac{z^*}{2} + a^\dagger)U(\theta) \\
\partial_{z^*} U(\theta) &= (\frac{z}{2} - a)U(\theta)
\end{aligned}$$

Then, we have

$$\partial_1 U(\theta) = \frac{\partial}{\partial\theta^1}U(\theta) = \frac{1}{\lambda}(\partial_z U(\theta) + \partial_{z^*} U(\theta)) = \frac{1}{\lambda}\{-a + a^\dagger + \frac{1}{2}(z - z^*)\}U(\theta) \tag{E.2}$$

$$\partial_2 U(\theta) = \frac{\partial}{\partial\theta^2}U(\theta) = \frac{\mathrm{i}}{\lambda}(-\partial_z U(\theta) + \partial_{z^*} U(\theta)) = \frac{\mathrm{i}}{\lambda}\{-a - a^\dagger + \frac{1}{2}(z + z^*)\}U(\theta). \tag{E.3}$$

$$
\begin{aligned}
|\psi\rangle &= U(\theta)\,|0\rangle_a\,|0\rangle_b = \mathrm{e}^{za^\dagger - z^* a}\,|0\rangle_a\,|0\rangle_b = |z\rangle_a\,|0\rangle_b \\
|\partial_1\psi\rangle &= \partial_1 U(\theta)\,|0\rangle_a\,|0\rangle_b = \frac{1}{\lambda}\{-a + a^\dagger + \frac{1}{2}(z - z^*)\}\,|z\rangle_a\,|0\rangle_b = \frac{1}{\lambda}\{a^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_a\,|0\rangle_b \\
\langle\psi|\partial_1\psi\rangle &= {}_a\langle z|\,{}_b\langle 0|\,\frac{1}{\lambda}\{a^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_a\,|0\rangle_b = \frac{1}{2\lambda}(-z + z^*) \\
\langle\partial_1\psi|\partial_1\psi\rangle &= {}_a\langle z|\,{}_b\langle 0|\,\frac{1}{\lambda^2}\{a - \frac{1}{2}(z + z^*)\}\{a^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_a\,|0\rangle_b \\
&= {}_a\langle z|\,\frac{1}{\lambda^2}\{a - \frac{1}{2}(z + z^*)\}\{a^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_a\,\langle 0|0\rangle_b \\
&= {}_a\langle z|\,\frac{1}{\lambda^2}\{a - \frac{1}{2}(z + z^*)\}\{a^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_a \\
&= {}_a\langle z|\,\frac{1}{\lambda^2}\{aa^\dagger - \frac{1}{2}(z + z^*)(a + a^\dagger) + \frac{1}{4}(z + z^*)^2\}\,|z\rangle_a \\
&= {}_a\langle z|\,\frac{1}{\lambda^2}\{a^\dagger a + 1 - \frac{1}{2}(z + z^*)(a + a^\dagger) + \frac{1}{4}(z + z^*)^2\}\,|z\rangle_a \\
&= \frac{1}{\lambda^2}\{|z|^2 + 1 - \frac{1}{2}(z + z^*)^2 + \frac{1}{4}(z + z^*)^2\} \\
&= \frac{1}{\lambda^2}\{|z|^2 + 1 - \frac{1}{4}(z + z^*)^2\} \\
g^R_{11} &= 4\,\langle\partial_1\psi|\partial_1\psi\rangle + 4\,\langle\psi|\partial_1\psi\rangle\,\langle\psi|\partial_1\psi\rangle \\
&= \frac{4}{\lambda^2}\{|z|^2 + 1 - \frac{1}{4}\{(z + z^*)^2 - (z - z^*)^2\} \\
&= \frac{4}{\lambda^2}\{|z|^2 + 1 - |z|^2\} \\
&= \frac{4}{\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
|\partial_2\psi\rangle &= \frac{\mathrm{i}}{\lambda}\{-a-a^\dagger+\frac{1}{2}(z+z^*)\}\,|z\rangle_a\,|0\rangle_b \\
&= \frac{\mathrm{i}}{\lambda}\{-a^\dagger-\frac{1}{2}(z-z^*)\}\,|z\rangle_a\,|0\rangle_b \\
&= \frac{1}{\mathrm{i}\lambda}\{a^\dagger+\frac{1}{2}(z-z^*)\}\,|z\rangle_a\,|0\rangle_b \\
\langle\partial_2\psi| &= -\frac{1}{\mathrm{i}\lambda}\{a-\frac{1}{2}(z-z^*)\}\,|z\rangle_a\,|0\rangle_b \\
\langle\partial_2\psi|\partial_1\psi\rangle &= {}_a\langle z|\,{}_b\langle 0|\,(-\frac{1}{\mathrm{i}\lambda})\{a-\frac{1}{2}(z-z^*)\}\frac{1}{\lambda}\{a^\dagger-\frac{1}{2}(z+z^*)\}\,|z\rangle_a\,|0\rangle_b \\
&= {}_a\langle z|\,\frac{\mathrm{i}}{\lambda^2}\{a-\frac{1}{2}(z-z^*)\}\{a^\dagger-\frac{1}{2}(z+z^*)\}\,|z\rangle_a\,\langle 0|0\rangle_b \\
&= {}_a\langle z|\,\frac{\mathrm{i}}{\lambda^2}\{a-\frac{1}{2}(z-z^*)\}\{a^\dagger-\frac{1}{2}(z+z^*)\}\,|z\rangle_a \\
&= {}_a\langle z|\,\frac{\mathrm{i}}{\lambda^2}\{aa^\dagger-\frac{1}{2}(z-z^*)a^\dagger-\frac{1}{2}(z+z^*)a+\frac{1}{2}(z-z^*)(z+z^*)\}\,|z\rangle_a \\
&= {}_a\langle z|\,\frac{\mathrm{i}}{\lambda^2}\{a^\dagger a+1-\frac{1}{2}(z-z^*)a^\dagger-\frac{1}{2}(z+z^*)a+\frac{1}{4}(z-z^*)(z+z^*)\}\,|z\rangle_a \\
&= \frac{\mathrm{i}}{\lambda^2}\{|z|^2+1-\frac{1}{2}(z-z^*)z^*-\frac{1}{2}(z+z^*)z+\frac{1}{4}(z^2-(z^*)^2\} \\
&= \frac{\mathrm{i}}{\lambda^2}\{|z|^2+1-\frac{1}{2}(|z|^2-(z^*)^2)-\frac{1}{2}(z^2+|z|^2)+\frac{1}{4}(z^2-(z^*)^2\} \\
&= \frac{\mathrm{i}}{\lambda^2}\{1-\frac{1}{2}(z^2-(z^*)^2)+\frac{1}{4}(z^2-(z^*)^2\} \\
&= \frac{\mathrm{i}}{\lambda^2}\{1-\frac{1}{4}(z^2-(z^*)^2)\} \\
\langle\psi|\partial_2\psi\rangle &= {}_a\langle z|\,{}_b\langle 0|\,\frac{1}{\mathrm{i}\lambda}\{a^\dagger+\frac{1}{2}(z-z^*)\}\,|z\rangle_a\,|0\rangle_b \\
&= \frac{1}{\mathrm{i}\lambda}\{z^*+\frac{1}{2}(z-z^*)\} \\
&= \frac{1}{2\mathrm{i}\lambda}(z+z^*) \\
g_{12} &= 4\,\langle\partial_2\psi|\partial_1\psi\rangle+4\,\langle\psi|\partial_2\psi\rangle\,\langle\psi|\partial_1\psi\rangle \\
&= \frac{4\mathrm{i}}{\lambda^2}\{1-\frac{1}{4}(z^2-(z^*)^2)\}+4\frac{4}{2\mathrm{i}\lambda}(z+z^*)\frac{1}{2\lambda}(-z+z^*) \\
&= \frac{4\mathrm{i}}{\lambda^2}\{1-\frac{1}{4}(z^2-(z^*)^2)\}+\frac{\mathrm{i}}{\lambda^2}(z+z^*)(z-z^*) \\
&= \frac{4\mathrm{i}}{\lambda^2}
\end{aligned}
$$

For Model 1 with the reference state $\rho_0^{(0)}$

$$
\begin{aligned}
G^R &= \frac{4}{\lambda^2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \\
G^S &= \frac{4}{\lambda^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(G^S)^{-1} &= \frac{\lambda^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{E.4}
$$

Since $\tilde{G^R} \neq Z$, we try the SLD defined below for the pure state $\rho_{\theta,a}$ to see if the result changes. For $\rho_{\theta,a} = |\psi_\theta\rangle\langle\psi_\theta|$,

$$
\begin{aligned}
\partial_i\rho_\theta &= |\partial_i\psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_i\psi_\theta| \\
\partial_i\rho_\theta &= \partial_i(\rho_\theta^2) = (\partial_i\rho_\theta)\rho_\theta + \rho_\theta(\partial_i\rho_\theta) = \frac{1}{2}\{2\partial_i\rho_\theta, \rho_\theta\}
\end{aligned}
$$

We regard $2\partial_i\rho_\theta$ as SLD $L^S_{\theta,i}$. From (E.2) and (E.3),

$$
\begin{aligned}
\partial_1 U(\theta) &= \frac{1}{\lambda}\{-a + a^\dagger + \frac{1}{2}(z - z^*)\}U(\theta) \\
\partial_2 U(\theta) &= \frac{\mathrm{i}}{\lambda}\{-a - a^\dagger + \frac{1}{2}(z + z^*)\}U(\theta).
\end{aligned}
$$

$|\partial_1\psi\rangle$ and $|\partial_1\psi\rangle$ are

$$
\begin{aligned}
|\partial_1\psi\rangle &= \frac{1}{\lambda}\{-a + a^\dagger + \frac{1}{2}(z - z^*)\}|z\rangle_a|0\rangle_b = \frac{1}{\lambda}\{-\frac{1}{2}z + a^\dagger + \frac{1}{2}(z - z^*)\}|z\rangle_a|0\rangle_b = \frac{1}{\lambda}\{a^\dagger - \frac{1}{2}(z + z^*)\}|z\rangle_a|0\rangle_b \\
|\partial_2\psi\rangle &= \frac{\mathrm{i}}{\lambda}\{-a - a^\dagger + \frac{1}{2}(z + z^*)\}|z\rangle_a|0\rangle_b = \frac{\mathrm{i}}{\lambda}\{-\frac{1}{2}z - a^\dagger + \frac{1}{2}(z + z^*)\}|z\rangle_a|0\rangle_b = \frac{\mathrm{i}}{\lambda}\{-a^\dagger - \frac{1}{2}(z - z^*)\}|z\rangle_a|z\rangle_b
\end{aligned}
$$

$L^S_{\theta,1}$ and $L^S_{\theta,2}$ are

$$
\begin{aligned}
L^S_{\theta,1} &= 2\partial_1\rho_\theta = \frac{2}{\lambda}[\{a^\dagger - \frac{1}{2}(z + z^*)\}|z\rangle_a\langle z| + |z\rangle_a\langle z|\{a - \frac{1}{2}(z + z^*)\}] \otimes |0\rangle_b\langle 0| \\
L^S_{\theta,2} &= 2\partial_2\rho_\theta = \frac{2\mathrm{i}}{\lambda}[\{-a^\dagger - \frac{1}{2}(z - z^*)\}|z\rangle_a\langle z| - |z\rangle_a\langle z|\{-a + \frac{1}{2}(z - z^*)\}] \otimes |0\rangle_b\langle 0|
\end{aligned}
$$

We define $\tilde{G^S}$ by $[\tilde{G^S}]_{ij} = \tilde{g^S}_{ij} = \mathrm{tr}\,[\rho_\theta L^S_{\theta,j} L^S_{\theta,i}]$.

$$\begin{aligned}
\tilde{g^S}_{11} &= \mathrm{tr}\,[\rho_\theta L^S_{\theta,1} L^S_{\theta,1}] \\
&= \frac{4}{\lambda^2}\, {}_a\langle z|\,\{a^\dagger - \frac{1}{2}(z+z^*)\}\,|z\rangle_a \langle z| + |z\rangle_a \langle z|\,\{a - \frac{1}{2}(z+z^*)\}\{a^\dagger - \frac{1}{2}(z+z^*)\}\,|z\rangle_a \langle z| + |z\rangle_a \langle z|\,\{a - \frac{1}{2}(z+z^*)\}\,|z\rangle_a \\
&= \frac{4}{\lambda^2}\, {}_a\langle z|\,\{|z\rangle_a \langle z|\,\{a - \frac{1}{2}(z+z^*) + \frac{1}{2}(z^*-z)\}\{a^\dagger - \frac{1}{2}(z+z^*) + \frac{1}{2}(z-z^*)\}\,|z\rangle_b \langle z|\}\,|z\rangle_b \\
&= \frac{4}{\lambda^2}\, {}_a\langle z|\,\{a - \frac{1}{2}(z+z^*) + \frac{1}{2}(z^*-z)\}\{a^\dagger - \frac{1}{2}(z+z^*) + \frac{1}{2}(z-z^*)\}\,|z\rangle_a \\
&= \frac{4}{\lambda^2}\, {}_a\langle z|\,(a-z)(a^\dagger - z^*)\,|z\rangle_a \\
&= \frac{4}{\lambda^2}\, {}_a\langle z|\,aa^\dagger - z^*a - za^\dagger + |z|^2\,|z\rangle_a \\
&= \frac{4}{\lambda^2}(|z|^2 + 1 - |z|^2 - |z|^2 + |z|^2) \\
&= \frac{4}{\lambda^2} \\
\tilde{g^S}_{12} &= \mathrm{tr}\,[\rho_\theta L^S_{\theta,2} L^S_{\theta,1}] \\
&= \frac{4\mathrm{i}}{\lambda^2}\, {}_a\langle z|\,\{-a^\dagger - \frac{1}{2}(z-z^*)\}\,|z\rangle_a \langle z| - |z\rangle_a \langle z|\,\{-a + \frac{1}{2}(z-z^*)\}(a^\dagger - z^*)\,|z\rangle_a \langle z|\}\,|z\rangle_a \\
&= \frac{4\mathrm{i}}{\lambda^2}\, {}_a\langle z|\,\{-z^* - \frac{1}{2}(z-z^*)\}\,|z\rangle_a \langle z| - |z\rangle_a \langle z|\,\{-a + \frac{1}{2}(z-z^*)\}(a^\dagger - z^*)\,|z\rangle_a \\
&= \frac{4\mathrm{i}}{\lambda^2}\, {}_a\langle z|\,\{|z\rangle_a \langle z|\,\{a - \frac{1}{2}(z-z^*) - \frac{1}{2}(z+z^*)\}(a^\dagger - z^*)\,|z\rangle_a \langle z|\}\,|z\rangle_a \\
&= \frac{4\mathrm{i}}{\lambda^2}\, {}_a\langle z|\,(a-z)(a^\dagger - z^*)\,|z\rangle_a \\
&= \frac{4\mathrm{i}}{\lambda^2}\, {}_a\langle z|\,(aa^\dagger - z^*a - z^*a^\dagger + |z|^2\,|z\rangle_a \\
&= \frac{4\mathrm{i}}{\lambda^2}(|z|^2 + 1 - |z|^2 - |z|^2 + |z|^2) \\
&= \frac{4\mathrm{i}}{\lambda^2}
\end{aligned}$$

$\tilde{G^S}$ defined by $[\tilde{G^S}]_{ij} = \tilde{g^S}_{ij}$ is

$$\tilde{G^S} = \frac{4}{\lambda^2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \tilde{G^R}$$

Then, $Z(\theta)$ is

$$Z(\theta) = (G^S)^{-1}\tilde{G^S}(G^S)^{-1} = \frac{\lambda^2}{4}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}$$

E.1.2 Model 2

$\rho_0^{(0)} = |0\rangle_a \langle 0| \otimes |0\rangle_b \langle 0|$ The unitary transformation is

$$\begin{aligned} U(\theta) &= \mathrm{e}^{-\mathrm{i}p_x\theta^1 - \mathrm{i}p_y\theta^1} \\ &= \mathrm{e}^{z_a a^\dagger - z_a^* a}\mathrm{e}^{z_b b^\dagger - z_b^* b} \end{aligned}$$

where $z_a = \frac{1}{\lambda}(\theta^1 - \mathrm{i}\theta^2)$ and $z_b = \frac{1}{\lambda}(\theta^1 + \mathrm{i}\theta^2) = z_a^*$.

$$|\psi_\theta\rangle = U(\theta)\,|0\rangle_a\,|0\rangle_b = |z_a\rangle_a\,|z_b\rangle_b \tag{E.5}$$

$$\begin{aligned} |\partial_1\psi_\theta\rangle &= \partial_1 U(\theta)\,|0\rangle_a\,|0\rangle_b = -\mathrm{i}p_x U(\theta)\,|0\rangle_a\,|0\rangle_b = -\mathrm{i}p_x\,|\psi_\theta\rangle = -\frac{1}{\lambda}\{(a^\dagger - a) + (b^\dagger - b)\}\,|\psi_\theta\rangle \\ |\partial_2\psi_\theta\rangle &= \partial_1 U(\theta)\,|0\rangle_a\,|0\rangle_b = -\mathrm{i}p_y U(\theta)\,|0\rangle_a\,|0\rangle_b = -\mathrm{i}p_y\,|\psi_\theta\rangle = -\frac{\mathrm{i}}{\lambda}\{(a^\dagger + a) - (b^\dagger + b)\}\,|\psi_\theta\rangle \end{aligned}$$

$$\begin{aligned} g_{11} &= 4\,\langle\partial_1\psi_\theta|\partial_1\psi_\theta\rangle + \langle\psi_\theta|\partial_1\psi_\theta\rangle \\ &= 4(\mathrm{i})(-\mathrm{i})\,\langle\psi_\theta|p_x^2|\psi_\theta\rangle - 4\,\langle\psi_\theta|p_x|\psi_\theta\rangle\,\langle\psi_\theta|p_x|\psi_\theta\rangle \\ &= 4\,\langle\psi_\theta|p_x^2|\psi_\theta\rangle - 4\,\langle\psi_\theta|p_x|\psi_\theta\rangle\,\langle\psi_\theta|p_x|\psi_\theta\rangle \\ g_{22} &= 4\,\langle\psi_\theta|p_y^2|\psi_\theta\rangle - 4\,\langle\psi_\theta|p_y|\psi_\theta\rangle\,\langle\psi_\theta|p_y|\psi_\theta\rangle \\ g_{12} &= 4\,\langle\psi_\theta|p_y p_x|\psi_\theta\rangle - 4\,\langle\psi_\theta|p_y|\psi_\theta\rangle\,\langle\psi_\theta|p_x|\psi_\theta\rangle \\ g_{21} &= 4\,\langle\psi_\theta|p_x p_y|\psi_\theta\rangle - 4\,\langle\psi_\theta|p_x|\psi_\theta\rangle\,\langle\psi_\theta|p_y|\psi_\theta\rangle \end{aligned}$$

$$\begin{aligned} \langle\psi_\theta|p_x|\psi_\theta\rangle &= \frac{\mathrm{i}}{2\lambda}\,\langle z_a, z_b|\,(a^\dagger - a) + (b^\dagger - b)\,|z_a, z_b\rangle \\ &= \frac{\mathrm{i}}{2\lambda}\{(z_a^* - z_a) + (z_b^* - z_b)\} = \frac{\mathrm{i}}{2\lambda}(-\mathrm{i})(\eta_a + \eta_b) = \frac{1}{2\lambda}(\eta_a + \eta_b) \\ \langle\psi_\theta|p_x^2|\psi_\theta\rangle &= -\frac{1}{4\lambda^2}\,\langle z_a, z_b|\,\{(a^\dagger - a) + (b^\dagger - b)\}^2\,|z_a, z_b\rangle \\ &= -\frac{1}{4\lambda^2}\,\langle z_a|\,(a^\dagger - a)^2\,|z_a\rangle + 2\,\langle z_a|\,(a^\dagger - a)\,|z_a\rangle\,\langle z_b|\,(b^\dagger - b)\,|z_b\rangle + \langle z_b|\,(b^\dagger - b)^2\,|z_b\rangle \\ &= -\frac{1}{4\lambda^2}\{(z_a^* - z_a)^2 + 1 + 2(z_a^* - z_a)(z_b^* - z_b) + (z_b^* - z_b)^2 + 1\} \\ &= -\frac{1}{4\lambda^2}(-\mathrm{i})^2(\eta_a^2 + 2\eta_a\eta_b + \eta_b^2 + 2) \\ &= \frac{1}{4\lambda^2}(\eta_a^2 + 2\eta_a\eta_b + \eta_b^2 + 2) \end{aligned}$$

where $\eta_a = 2\mathrm{Im}(z_a)$, $\eta_b = 2\mathrm{Im}(z_b)$.

$$\begin{aligned} \langle\psi_\theta|p_y|\psi_\theta\rangle &= \frac{1}{2\lambda}(\xi_a - \xi_b) \\ \langle\psi_\theta|p_y^2|\psi_\theta\rangle &= -\frac{1}{4\lambda^2}\,\langle z_a, z_b|\,\{(a^\dagger + a) - (b^\dagger + b)\}^2\,|z_a, z_b\rangle = -\frac{1}{4\lambda^2}(\xi_a^2 - 2\xi_a\xi_b + \xi_b^2 + 2) \end{aligned}$$

where $\xi_a = 2\mathrm{Re}(z_a)$, $\xi_b = 2\mathrm{Re}(z_b)$.

$$
\begin{aligned}
\langle\psi_\theta|p_x p_y|\psi_\theta\rangle &= \frac{\mathrm{i}}{4\lambda^2}\langle z_a, z_b|\{(a^\dagger + a) - (b^\dagger + b)\}\{(a^\dagger - a) + (b^\dagger - b)\}|z_a, z_b\rangle \\
&= \frac{1}{4\lambda^2}\langle z_a|(a^\dagger + a)(a^\dagger - a)|z_a\rangle + \langle z_a|(a^\dagger + a)|z_a\rangle\langle z_b|(b^\dagger - b)|z_b\rangle \\
&- \langle z_a|(a^\dagger - a)|z_a\rangle\langle z_b|(b^\dagger + b)|z_b\rangle - \langle z_b|(b^\dagger + b)(b^\dagger - b)|z_b\rangle \\
&= \frac{1}{4\lambda^2}\{\xi_a\eta_a + 1 + \xi_a\eta_b - \xi_b\eta_a - (\xi_b\xi_b + 1)\} \\
&= \frac{1}{4\lambda^2}(\xi_a\eta_a + \xi_a\eta_b - \xi_b\eta_a - \xi_b\xi_b) \\
&= \frac{1}{4\lambda^2}\{\xi_a(\eta_a + \eta_b) - \xi_b(\eta_a + \eta_b)\} \\
&= \frac{1}{4\lambda^2}\{(\xi_a - \xi_b)(\eta_a + \eta_b)\}
\end{aligned}
$$

$$
\begin{aligned}
g_{11} &= 4\langle\psi_\theta|p_x^2|\psi_\theta\rangle - 4\langle\psi_\theta|p_x|\psi_\theta\rangle\langle\psi_\theta|p_x|\psi_\theta\rangle = \frac{1}{\lambda^2}\{(\eta_a^2 + 2\eta_a\eta_b + \eta_b^2 + 2) - (\eta_a + \eta_b)^2\} = \frac{2}{\lambda^2} \\
g_{22} &= 4\langle\psi_\theta|p_y^2|\psi_\theta\rangle - 4\langle\psi_\theta|p_y|\psi_\theta\rangle\langle\psi_\theta|p_y|\psi_\theta\rangle = \frac{1}{\lambda^2}\{(\xi_a^2 - 2\xi_a\xi_b + \xi_b^2 + 2) - (\xi_a - \xi_b)^2\} = \frac{2}{\lambda^2} \\
g_{12} &= 4\langle\psi_\theta|p_x p_y|\psi_\theta\rangle - 4\langle\psi_\theta|p_y|\psi_\theta\rangle\langle\psi_\theta|p_x|\psi_\theta\rangle = \frac{1}{\lambda^2}\{(\xi_a - \xi_b)(\eta_a + \eta_b) - (\eta_a + \eta_b)(\xi_a - \xi_b)\} = 0
\end{aligned}
$$

For Model 2 with the reference state $\rho_0^{(0)}$,

$$
\begin{aligned}
G^S &= \frac{2}{\lambda^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(G^S)^{-1} &= \frac{\lambda^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{E.6}
$$

E.2 Reference state 1

E.2.1 Model 1

Reference state 1 : $\rho_0^{(1)} = \sum f(n)\,|n\rangle_a\langle n| \otimes |0\rangle_b\langle 0|$

$$
U(\theta) = e^{-\mathrm{i}\pi_x\theta^1 - \mathrm{i}\pi_y\theta^2} = e^{za^\dagger - z^* a} \tag{E.7}
$$

Since $U(\theta)$ has a, $a^\dagger$ only, it gives no effect on $|0\rangle_b \langle 0|$.

$$
(G^R)^{-1} = \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa^2 \end{pmatrix}
$$
$$
(G^S)^{-1} = \frac{\lambda^2}{4}\begin{pmatrix} 1+4\kappa^2 & 0 \\ 0 & 1+4\kappa^2 \end{pmatrix}
\tag{E.8}
$$

E.2.2 Model 2

Reference state 1 : $\rho_0^{(1)} = \sum f(n)\,|n\rangle_a \langle n| \otimes |0\rangle_b \langle 0|$

No change in G_a^R or G_a^S. We will derive SLD for $\rho_{\theta,b}$. For $\rho_{\theta,b} = |\psi_\theta\rangle\langle\psi_\theta|$,

$$
\begin{aligned}
\partial_i\rho_\theta &= |\partial_i\psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_i\psi_\theta| \\
\partial_i\rho_\theta &= \partial_i(\rho_\theta^2) = (\partial_i\rho_\theta)\rho_\theta + \rho_\theta(\partial_i\rho_\theta) = \frac{1}{2}\{2\partial_i\rho_\theta, \rho_\theta\}
\end{aligned}
$$

We regard $2\partial_i\rho_\theta$ as SLD $L_{\theta,i}^S$. The unitary transformation is

$$
U(\theta) = U_a(z_a) \otimes U_b(z_b) \tag{E.9}
$$

where $U_a(z_a) = \mathrm{e}^{z_a a^\dagger - z_a^* a}$ and $U_b(z_b) = \mathrm{e}^{z_b b^\dagger - z_b^* b}$. We now consider 'b' only. Hereafter we write z_b as z. Therefore, $z = \frac{1}{2\lambda}(\theta^1 + \mathrm{i}\theta^2)$.

$$
\begin{aligned}
\frac{\partial U_b}{\partial\theta^1} &= \frac{1}{2\lambda}\left(\frac{\partial z}{\partial\theta^1}\frac{\partial U_b}{\partial z} + \frac{\partial z^*}{\partial\theta^1}\frac{\partial U_b}{\partial z^*}\right) = \frac{1}{2\lambda}\left(\frac{\partial U_b}{\partial z} + \frac{\partial U_b}{\partial z^*}\right) \\
\frac{\partial U_b}{\partial\theta_2} &= \frac{\mathrm{i}}{2\lambda}\left(\frac{\partial z}{\partial\theta_2}\frac{\partial U_b}{\partial z} + \frac{\partial z^*}{\partial\theta_2}\frac{\partial U_b}{\partial z^*}\right) = \frac{\mathrm{i}}{2\lambda}\left(\frac{\partial U_b}{\partial z} + \frac{\partial U_b}{\partial z^*}\right) \\
\frac{\partial U_b}{\partial z} &= \left(-\frac{1}{2}z^* + b^\dagger\right)U_b \\
\frac{\partial U_b}{\partial z^*} &= \left(\frac{1}{2}z - b\right)U_b \\
\because\ U_b(z_b) &= \mathrm{e}^{z_b b^\dagger - z_b^* b} = \mathrm{e}^{-\frac{|z|^2}{2\kappa^2}}\mathrm{e}^{z b^\dagger}\mathrm{e}^{-z^* b} = \mathrm{e}^{\frac{|z|^2}{2\kappa^2}}\mathrm{e}^{-z^* b}\mathrm{e}^{z b^\dagger}
\end{aligned}
\tag{E.10}
$$

Then, $\frac{\partial U_b}{\partial\theta^1}$ and $\frac{\partial U_b}{\partial\theta_2}$ are

$$
\begin{aligned}
\frac{\partial U_b}{\partial\theta^1} &= \frac{1}{2\lambda}\left(\frac{1}{2}z - \frac{1}{2}z^* + b^\dagger - b\right)U_b \\
\frac{\partial U_b}{\partial\theta^2} &= \frac{\mathrm{i}}{2\lambda}\left(\frac{1}{2}z - \frac{1}{2}z^* + b^\dagger - b\right)U_b
\end{aligned}
$$

$|\partial_1\psi\rangle$ and $|\partial_1\psi\rangle$ are

$$
\begin{aligned}
|\partial_1\psi\rangle &= \frac{1}{2\lambda}(\frac{1}{2}z - \frac{1}{2}z^* + b^\dagger - b)\,|z\rangle_b = \frac{1}{2\lambda}(\frac{1}{2}z - z - \frac{1}{2}z^* + b^\dagger)\,|z\rangle_b = \frac{1}{2\lambda}\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b \\
|\partial_2\psi\rangle &= \frac{\mathrm{i}}{2\lambda}(-\frac{1}{2}z^* + \frac{1}{2}z + b^\dagger - b)\,|z\rangle_b = \frac{\mathrm{i}}{2\lambda}\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b
\end{aligned}
\tag{E.11}
$$

$L^S_{\theta,1}$ and $L^S_{\theta,2}$ are

$$
\begin{aligned}
L^S_{\theta,1} &= 2\partial_1\rho_\theta = \frac{1}{\lambda}[\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| + |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\} \\
L^S_{\theta,2} &= 2\partial_2\rho_\theta = \frac{\mathrm{i}}{\lambda}[\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| - |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\}
\end{aligned}
\tag{E.12}
$$

$\tilde{G^S}$ is defined as $[\tilde{G^S}]_{ij} = \mathrm{tr}[\rho_\theta L^S_{\theta,1} L^S_{\theta,1}]$.

Then, SLD Fisher information G^S is $G^S = \mathrm{Re}\,\tilde{G^S}$. First, we evaluate $\tilde{G^S}$.

$$
\begin{aligned}
\tilde{g^s}_{1\,1} &= \mathrm{tr}\,[\rho_\theta L^S_{\theta,1} L^S_{\theta,1}] \\
&= \frac{1}{\lambda^2}\,{}_b\langle z|\,\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| + |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\}\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| + |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\}\,|z\rangle_b \\
&= \frac{1}{\lambda^2}\,{}_b\langle z|\,\{|z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*) + \frac{1}{2}(z^* - z)\}\{b^\dagger - \frac{1}{2}(z + z^*) + \frac{1}{2}(z - z^*)\}\,|z\rangle_b\,\langle z|\}\,|z\rangle_b \\
&= \frac{1}{\lambda^2}\,{}_b\langle z|\,\{b - \frac{1}{2}(z + z^*) + \frac{1}{2}(z^* - z)\}\{b^\dagger - \frac{1}{2}(z + z^*) + \frac{1}{2}(z - z^*)\}\,|z\rangle_b \\
&= \frac{1}{\lambda^2}\,{}_b\langle z|\,(b - z)(b^\dagger - z^*)\,|z\rangle_b \\
&= \frac{1}{\lambda^2}(b - z)(|z|^2 + 1 - |z|^2 - |z|^2 + |z|^2) \\
&= \frac{1}{\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g^s}_{1\,2} &= \mathrm{tr}\,[\rho_\theta L^S_{\theta,2} L^S_{\theta,1}] \\
&= \frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| - |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\}\{b^\dagger - \frac{1}{2}(z + z^*)\}\,|z\rangle_b\,\langle z| + |z\rangle_b\,\langle z|\,\{b - \frac{1}{2}(z + z^*)\}\,|z\rangle_b \\
&= \frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,\{|z\rangle_b\,\langle z|\,\{-b + \frac{1}{2}(z + z^*) + \frac{1}{2}(z - z^*)\}(b^\dagger - z^*)\,|z\rangle_b \\
&= \frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,(-b + z^*)(b^\dagger - z^*)\,|z\rangle_b \\
&= -\frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,(b - z^*)(b^\dagger - z^*)\,|z\rangle_b \\
&= -\frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,\{bb^\dagger - z^*b - z^*b^\dagger + (z^*)^2\}\,|z\rangle_b \\
&= -\frac{\mathrm{i}}{\lambda^2}\,{}_b\langle z|\,\{|z|^2 + 1 - |z|^2 - (z^*)^2 + (z^*)^2\}\,|z\rangle_b \\
&= -\frac{\mathrm{i}}{\lambda^2}
\end{aligned}
$$

G_b^S is

$$G_b^S = \frac{\mathrm{i}}{\lambda^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$\tilde{G_b^S}$ is

$$\tilde{G_b^S} = \frac{\mathrm{i}}{\lambda^2}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}$$

From $G^S = G_a^S + G_b^S$,

$$G^S = \frac{1}{\lambda^2}\frac{1}{1+4\kappa_a^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\lambda^2}\begin{pmatrix}\frac{1}{1+4\kappa_a^2}+1 & 0\\ 0 & \frac{1}{1+4\kappa_a^2}+1\end{pmatrix} = \frac{2}{\lambda^2}\frac{1+2\kappa_a^2}{1+4\kappa_a^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

The inverse of G^S is

$$(G^S)^{-1} = \frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{E.13}$$

Next, we obtain $\tilde{G_a^S}$ to determine $\tilde{G^S}$.

$$\begin{aligned}
L_{0,1\,a}^S &= \frac{1}{\lambda(1+4\kappa_a^2)}(a^\dagger + a)\\
L_{0,2\,a}^S &= -\frac{\mathrm{i}}{\lambda(1+4\kappa_a^2)}(a^\dagger - a)
\end{aligned} \tag{E.14}$$

First, we calculate $\tilde{g}_{a,1\,1}$.

$$\begin{aligned}
\tilde{g}_{a,1\,1} &= \mathrm{tr}\,[\rho_0 L_{0,1\,a}^S L_{0,1\,a}^S]\\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\langle z|\,(a^\dagger + a)(a^\dagger + a)\,|z\rangle\, d^2z\\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\langle z|\,(a^\dagger)^2 + a^\dagger a + aa^\dagger + a^2)\,|z\rangle\, d^2z\\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\langle z|\,(a^\dagger)^2 + 2a^\dagger a + a^2 + 1\,|z\rangle\, d^2z\\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}(1+4\kappa_a^2)\\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)}
\end{aligned} \tag{E.15}$$

$$
\begin{aligned}
\tilde{g}_{a,12} &= \mathrm{tr}\,[\rho_0 L^S_{0,2\,a} L^S_{0,1\,a}] \\
&= -\frac{\mathrm{i}}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\,\langle z|\,(a^\dagger - a)(a^\dagger + a)\,|z\rangle\, d^2z \\
&= -\frac{\mathrm{i}}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\,\langle z|\,(a^\dagger)^2 + a^\dagger a - aa^\dagger - a^2)\,|z\rangle\, d^2z \\
&= -\frac{\mathrm{i}}{\lambda^2(1+4\kappa_a^2)^2}\frac{1}{2\pi\kappa_a^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa_a^2}}\,\langle z|\,(a^\dagger)^2 - a^2 - 1\,|z\rangle\, d^2z \\
&= -\frac{\mathrm{i}}{\lambda^2(1+4\kappa_a^2)^2}(-1) \\
&= \frac{\mathrm{i}}{\lambda^2(1+4\kappa_a^2)^2}
\end{aligned}
\tag{E.16}
$$

$\tilde{G^S_a}$ is

$$
\tilde{G^S_a} = \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} 1+4\kappa_a^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa_a^2 \end{pmatrix}
$$

$\tilde{G^S}$ is

$$
\begin{aligned}
\tilde{G^S} = \tilde{G^S_a} + \tilde{G^S_b} &= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} 1+4\kappa_a^2 & \mathrm{i} \\ -\mathrm{i} & 1+4\kappa_a^2 \end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} 1+4\kappa_a^2+(1+4\kappa_a^2)^2 & \mathrm{i}\{1-(1+4\kappa_a^2)^2\} \\ -\mathrm{i}\{1-(1+4\kappa_a^2)^2\} & 1+4\kappa_a^2+(1+4\kappa_a^2)^2 \end{pmatrix} \\
&= \frac{1}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} (1+4\kappa_a^2)(2+4\kappa_a^2) & -\mathrm{i}\,4\kappa_a^2(2+4\kappa_a^2) \\ \mathrm{i}\,4\kappa_a^2(2+4\kappa_a^2) & (1+4\kappa_a^2)(2+4\kappa_a^2) \end{pmatrix} \\
&= \frac{2(1+2\kappa_a^2)}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} 1+4\kappa_a^2 & -\mathrm{i}\,4\kappa_a^2 \\ \mathrm{i}\,4\kappa_a^2 & 1+4\kappa_a^2 \end{pmatrix}
\end{aligned}
$$

Then, $Z(\theta)$ is

$$
\begin{aligned}
Z(\theta) = (G^S)^{-1}\tilde{G^S}(G^S)^{-1} &= (\frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2})^2\frac{2(1+2\kappa_a^2)}{\lambda^2(1+4\kappa_a^2)^2}\begin{pmatrix} 1+4\kappa_a^2 & -\mathrm{i}\,4\kappa_a^2 \\ \mathrm{i}\,4\kappa_a^2 & 1+4\kappa_a^2 \end{pmatrix} \\
&= \frac{\lambda^2}{2}\frac{1}{1+2\kappa_a^2}\begin{pmatrix} 1+4\kappa_a^2 & -\mathrm{i}\,4\kappa_a^2 \\ \mathrm{i}\,4\kappa_a^2 & 1+4\kappa_a^2 \end{pmatrix} = \frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix} 1 & -\mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} \\ \mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} & 1 \end{pmatrix}
\end{aligned}
$$

From (E.13),

$$
(G^S)^{-1} = \frac{\lambda^2}{2}\frac{1+4\kappa_a^2}{1+2\kappa_a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.17}
$$

If $(G^S)^{-1} \leq Z$, then

$$
\begin{aligned}
&\Leftrightarrow I \leq \begin{pmatrix} 1 & -\mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} \\ \mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} & 1 \end{pmatrix} \\
&\Leftrightarrow \begin{pmatrix} 1-1 & -\mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} \\ \mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} & 1-1 \end{pmatrix} \geq 0 \\
&\Leftrightarrow \begin{pmatrix} 0 & -\mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} \\ \mathrm{i}\,\frac{4\kappa_a^2}{1+4\kappa_a^2} & 0 \end{pmatrix} \geq 0
\end{aligned}
$$

But, this is not true, because the eigenvalues of the matrix above is $\pm\dfrac{4\kappa_a^2}{1+4\kappa_a^2}$. There is no matrix ordering between $(G^S)^{-1}$ and Z.

E.3 Reference state 2 Model 2

Reference state 2 : $\rho_0^{(2)} = \rho_{0\,a} \otimes \rho_{0\,b} \;\wedge\; \kappa_b \to 0$

$$
\begin{aligned}
(G^R)^{-1} &= \lambda^2 \frac{2\kappa_a^2}{1+4\kappa_a^2} \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \\
(G^S)^{-1} &= \frac{\lambda^2}{2} \frac{1+4\kappa_a^2}{1+2\kappa_a^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{E.18}
$$

Appendix F

Commutation relation of L_1^S and L_2^S : Model 2

As explained in Appedix D.6.5,

$$L_1^S = \frac{1}{1+4\kappa_a^2}(a+a^\dagger) + \frac{1}{1+4\kappa_b^2}(b+b^\dagger)$$
$$L_2^S = \frac{\mathrm{i}}{1+4\kappa_a^2}(a-a^\dagger) - \frac{1}{1+4\kappa_b^2}(b-b^\dagger)$$

a, $a^\dagger$ and b, $b^\dagger$ commute, we only need to compute the commutation relation, $[a+a^\dagger, a-a^\dagger]$ and $[b+b^\dagger, b-b^\dagger]$.

$$[L_1^S, L_2^S] = \frac{1}{1+4\kappa_a^2}[a+a^\dagger, a-a^\dagger] - \frac{1}{1+4\kappa_b^2}[b+b^\dagger, b-b^\dagger]$$
$$[a+a^\dagger, a-a^\dagger] = [b+b^\dagger, b-b^\dagger],$$

because $[a, a^\dagger] = [b, b^\dagger] = 1$.

$$[L_1^S, L_2^S] = \frac{1}{1+4\kappa_a^2} - \frac{1}{1+4\kappa_b^2}[a+a^\dagger, a-a^\dagger]$$

$[a+a^\dagger, a-a^\dagger]$ is

$$\begin{aligned}[a+a^\dagger, a-a^\dagger] &= [a, a-a^\dagger] + [a^\dagger, a-a^\dagger] \\ &= -[a, a^\dagger] + [a^\dagger, a] = -2\end{aligned}$$

Then, $[L_1^S, L_2^S]$ is

$$[L_1^S, L_2^S] = -2\{\frac{1}{1+4\kappa_a^2} - \frac{1}{1+4\kappa_b^2}\}$$
$$\therefore [L_1^S, L_2^S] = 0 \text{ only if } \kappa_a = \kappa_b \tag{F.1}$$

In particular, when $\kappa_a,\ \kappa_b \to 0$,

$$
\begin{aligned}
L_1^S &= (a + a^\dagger) + (b + b^\dagger) = \frac{2}{\lambda} x \\
L_2^S &= (a - a^\dagger) - (b - b^\dagger) = \frac{2}{\mathrm{i}\lambda} y
\end{aligned}
$$

Appendix G

Model 1 : Relativistic case

G.1 Hamiltonian

Hamiltonian of an electron in a uniform magnetic field is

$$
\begin{aligned}
H &= \sum_{i=1}^{3} \gamma^0 \gamma^i \Pi^i + m\gamma^0 \\
&= \gamma^0 \gamma^1 \Pi^1 + \gamma^0 \gamma^2 \Pi^2 + \gamma^0 \gamma^3 \Pi^3 + m\gamma^0
\end{aligned}
$$

where γ^μ, $(\mu = 0,\ 1,\ 2,\ 3)$ are the gamma matrices. In Dirac representation, they are

$$
\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},\ \ \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
$$

where σ_i and I are Pauli matrix and 2 x 2 identity matrix, respectively. Since the commutation relation between Π^1 and Π^2 is the same as non-relativistic case, they are

$$
\begin{aligned}
\Pi^1 &= \frac{1}{\lambda}(\alpha^\dagger + \alpha) \\
\Pi^2 &= \frac{\mathrm{i}}{\lambda}(\alpha^\dagger - \alpha)
\end{aligned}
$$

where $\alpha,\ \alpha^\dagger$ are the annihilation creation operators which satisfies $[\alpha,\ \alpha^\dagger] = 1$. With using $\alpha,\ \alpha^\dagger$, Hamiltonian is

$$
\begin{aligned}
H &= \frac{1}{\lambda}\{\gamma^0\gamma^1(\alpha^\dagger + \alpha) + \mathrm{i}\gamma^0\gamma^2(\alpha^\dagger - \alpha)\} + m\gamma^0 \\
&= \frac{1}{\lambda}\{\begin{pmatrix} 0 & \sigma_1 + \mathrm{i}\sigma_2 \\ \sigma_1 + \mathrm{i}\sigma_2 & 0 \end{pmatrix}\alpha^\dagger + \begin{pmatrix} 0 & \sigma_1 - \mathrm{i}\sigma_2 \\ \sigma_1 - \mathrm{i}\sigma_2 & 0 \end{pmatrix}\alpha\} + m\gamma^0 \\
&= \frac{1}{\lambda}\{\begin{pmatrix} 0 & \sigma^+ \\ \sigma^+ & 0 \end{pmatrix}\alpha^\dagger + \begin{pmatrix} 0 & \sigma^- \\ \sigma^- & 0 \end{pmatrix}\alpha\} + m\gamma^0 \\
&= \begin{pmatrix} m & 0 & 0 & \frac{1}{\lambda}\alpha^\dagger \\ 0 & m & \frac{1}{\lambda}\alpha & 0 \\ 0 & \frac{1}{\lambda}\alpha^\dagger & -m & 0 \\ \frac{1}{\lambda}\alpha & 0 & 0 & -m \end{pmatrix}
\end{aligned}
$$

To make H simpler, we define the unitary matrix U is defined as

$$
U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = U^\dagger.
$$

$UU^\dagger$ is

$$
UU^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = I
$$

The unitary transformation with U gives

$$UHU^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m & 0 & 0 & \frac{1}{\lambda}\alpha^{\dagger} \\ 0 & m & \frac{1}{\lambda}\alpha & 0 \\ 0 & \frac{1}{\lambda}\alpha^{\dagger} & -m & 0 \\ \frac{1}{\lambda}\alpha & 0 & 0 & -m \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m & 0 & 0 & \frac{1}{\lambda}\alpha^{\dagger} \\ \frac{1}{\lambda}\alpha & 0 & 0 & -m \\ 0 & \frac{1}{\lambda}\alpha^{\dagger} & -m & 0 \\ 0 & m & \frac{1}{\lambda}\alpha & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m & \frac{1}{\lambda}\alpha^{\dagger} & 0 & 0 \\ \frac{1}{\lambda}\alpha & -m & 0 & 0 \\ 0 & 0 & -m & \frac{1}{\lambda}\alpha^{\dagger} \\ 0 & 0 & \frac{1}{\lambda}\alpha & m \end{pmatrix}$$

$$H' = UHU^{\dagger} \tag{G.1}$$

G.2 Reference state

We choose the thermal state as a reference state. Therefore, the next step is to determine $e^{-\beta H}$. Hamiltonian is

$$H' = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

Therefore, we get

$$e^{-\beta H'} = \begin{pmatrix} e^{-\beta h_1} & 0 \\ 0 & e^{-\beta h_2} \end{pmatrix} \tag{G.2}$$

where

$$h_1 = \begin{pmatrix} m & \frac{1}{\lambda}\alpha^{\dagger} \\ \frac{1}{\lambda}\alpha & -m \end{pmatrix} = m\sigma_3 + \frac{1}{\lambda}\{\sigma_-\alpha^{\dagger} + \sigma_+\alpha\}$$

$$h_2 = \begin{pmatrix} -m & \frac{1}{\lambda}\alpha^{\dagger} \\ \frac{1}{\lambda}\alpha & m \end{pmatrix} = -m\sigma_3 + \frac{1}{\lambda}\{\sigma_-\alpha^{\dagger} + \sigma_+\alpha\}$$

We define projection operators, P_n ($n \geq 1$) and P_0 by

$$P_n = \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} |n\rangle\langle n| & 0 \\ 0 & |n-1\rangle\langle n-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix}, \quad P_0 = \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix}$$

The summation of P_n and P_0 is

$$\begin{aligned} P_0 + \sum_{n=1} P_n &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} \sum_{n=1} |n\rangle\langle n| + |0\rangle\langle 0| & 0 \\ 0 & \sum_{n=1} |n-1\rangle\langle n-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \\ &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \end{aligned}$$

h_1 multiplied by P_n from the left and $P_{n'}$ from the right gives

$$\begin{aligned} P_n h_1 P_{n'} &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} |n\rangle\langle n| & 0 \\ 0 & |n-1\rangle\langle n-1| \end{pmatrix} \begin{pmatrix} m & \frac{1}{\lambda}\alpha^\dagger \\ \frac{1}{\lambda}\alpha & -m \end{pmatrix} \begin{pmatrix} |n'\rangle\langle n'| & 0 \\ 0 & |n'-1\rangle\langle n'-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \\ &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} m|n\rangle\langle n| & \frac{1}{\lambda}|n\rangle\langle n|\alpha^\dagger \\ \frac{1}{\lambda}|n-1\rangle\langle n-1|\alpha & -m|n-1\rangle\langle n-1| \end{pmatrix} \begin{pmatrix} |n'\rangle\langle n'| & 0 \\ 0 & |n'-1\rangle\langle n'-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \\ &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} m|n\rangle\langle n|n'\rangle\langle n'| & \frac{1}{\lambda}|n\rangle\langle n|\alpha^\dagger|n'-1\rangle\langle n'-1| \\ \frac{1}{\lambda}|n-1\rangle\langle n-1|\alpha|n'\rangle\langle n'| & -m|n\rangle\langle n-1|n'-1\rangle\langle n'-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \\ &= \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} m|n\rangle\langle n|n'\rangle\langle n'| & \frac{\sqrt{n}}{\lambda}|n\rangle\langle n|n'\rangle\langle n'-1| \\ \frac{\sqrt{n'}}{\lambda}|n-1\rangle\langle n-1|n'-1\rangle\langle n'| & -m|n\rangle\langle n-1|n'-1\rangle\langle n'-1| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \\ &= \delta_{n,n'} \begin{pmatrix} |g\rangle & |e\rangle \end{pmatrix} \begin{pmatrix} m|n\rangle\langle n| & \frac{1}{\lambda}\sqrt{n}|n\rangle\langle n-1| \\ \frac{1}{\lambda}\sqrt{n}|n\rangle\langle n-1| & -m|n\rangle\langle n| \end{pmatrix} \begin{pmatrix} \langle g| \\ \langle e| \end{pmatrix} \end{aligned}$$

Therefore, $P_n h_1 P_n$ can be expressed as

$$P_n h_1 P_n = \begin{pmatrix} |g\rangle|n\rangle & |e\rangle|n-1\rangle \end{pmatrix} \begin{pmatrix} m & \frac{\sqrt{n}}{\lambda} \\ \frac{\sqrt{n}}{\lambda} & -m \end{pmatrix} \begin{pmatrix} \langle g|\langle n| \\ \langle e|\langle n-1| \end{pmatrix} \tag{G.3}$$

Define τ_2 and τ_3 as

$$\tau_1 = \begin{pmatrix} |g\rangle|n\rangle & |e\rangle|n-1\rangle \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle g|\langle n| \\ \langle e|\langle n-1| \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} |g\rangle|n\rangle & |e\rangle|n-1\rangle \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle g|\langle n| \\ \langle e|\langle n-1| \end{pmatrix}$$

and

$$P_n = \begin{pmatrix} |g\rangle\,|n\rangle & |e\rangle\,|n-1\rangle \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle g|\,\langle n| \\ \langle e|\,\langle n-1| \end{pmatrix}$$

Therefore,

$$h_{1,n} = P_n h_1 P_n = \frac{\sqrt{n}}{\lambda}\tau_1 + m\tau_3 = \vec{a_{1,n}}\,\vec{\tau} \tag{G.4}$$

where $\vec{a_1} = (\frac{\sqrt{n}}{\lambda}, 0, m)$

Define $P_{\pm}^{(n)}$ as follows.

$$P_{\pm}^{(n)} = \frac{1}{2}(P_n \pm \frac{\vec{a_{1,n}}}{|\vec{a_{1,n}}|}\vec{\tau}) \tag{G.5}$$

Then,

$$P_{+}^{(n)} + P_{-}^{(n)} = P_n \tag{G.6}$$

$$P_{+}^{(n)} - P_{-}^{(n)} = \frac{\vec{a_{1,n}}}{|\vec{a_{1,n}}|}\vec{\tau} \tag{G.7}$$

From (G.4),

$$h_{1,n} = |\vec{a_{1,n}}|\,(P_{+}^{(n)} - P_{-}^{(n)}) \tag{G.8}$$

$$\begin{aligned}
h_{1,n}P_{\pm}^{(n)} &= \frac{1}{2}\vec{a_{1,n}}\,\vec{\tau}\,(P_n \pm \frac{\vec{a_{1,n}}}{|\vec{a_{1,n}}|}\vec{\tau}) \\
&= \frac{1}{2}\vec{a_{1,n}}\,\vec{\tau} \pm \frac{1}{2}\frac{1}{|\vec{a_{1,n}}|}(\vec{a_{1,n}}\vec{\tau})^2 \\
&= \frac{1}{2}\vec{a_{1,n}}\,\vec{\tau} \pm \frac{1}{2}|\vec{a_{1,n}}|P_n \\
&= \pm\frac{1}{2}|\vec{a_{1,n}}|\,(P_n \pm \frac{\vec{a_{1,n}}}{|\vec{a_{1,n}}|}\vec{\tau}) \\
&= \pm|\vec{a_{1,n}}|\,P_{\pm}^{(n)}
\end{aligned}$$

$$\begin{aligned}
h_{1,n}P_n &= h_{1,n}(P_{+}^{(n)} + P_{-}^{(n)}) \\
&= a_{1,n}\,P_{+}^{(n)} - a_{1,n}\,P_{-}^{(n)}
\end{aligned}$$

where $a_{1,n} = \sqrt{\frac{n}{\lambda^2} + m^2}$.

$$
\begin{aligned}
\mathrm{e}^{-\beta h_1} &= (\sum_n P_n)\mathrm{e}^{-\beta h_1}(\sum_m P_m) \\
&= \sum_n P_n \mathrm{e}^{-\beta h_1} P_n \\
&= \sum_n \mathrm{e}^{-\beta h_{1,n}} \\
&= \sum_n \mathrm{e}^{-\beta a_{1,n}} P_+^{(n)} + \mathrm{e}^{\beta a_{1,n}} P_-^{(n)} \\
&= \sum_n \mathrm{e}^{-\beta a_{1,n}} \frac{1}{2}(P_n + \frac{\vec{a_{1,n}}}{a_{1,n}}\vec{\tau}) + \mathrm{e}^{\beta a_{1,n}} \frac{1}{2}(P_n - \frac{\vec{a_{1,n}}}{a_{1,n}}\vec{\tau}) \\
&= \sum_n \{\cosh(\beta a_{1,n}) P_n + \sinh(\beta a_{1,n}) \frac{\vec{a_{1,n}}}{a_{1,n}}\vec{\tau}\}
\end{aligned}
$$

For $n = 0$, the explicit expression of $\mathrm{e}^{-\beta h_{1,n}}$ with the current representation is

$$
\begin{aligned}
\mathrm{e}^{-\beta h_{1,n}} &= \begin{pmatrix} |g\rangle\,|0\rangle & 0 \end{pmatrix} \begin{pmatrix} \cosh(\beta a_{1,0}) + \frac{m}{a_{1,0}}\sinh(\beta a_{1,n}) & 0 \\ 0 & \cosh(\beta a_{1,0}) - \frac{m}{a_{1,0}}\sinh(\beta a_{1,0}) \end{pmatrix} \begin{pmatrix} \langle g|\,\langle 0| \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} |g\rangle\,|0\rangle\,\{\cosh(\beta a_{1,0}) + \frac{m}{a_{1,0}}\sinh(\beta a_{1,0})\} & 0 \end{pmatrix} \begin{pmatrix} \langle g|\,\langle 0| \\ 0 \end{pmatrix} \\
&= |g\rangle\,|0\rangle\,\langle g|\,\langle 0|\,\{\cosh(\beta a_{1,0}) + \frac{m}{a_{1,0}}\sinh(\beta a_{1,0})\}
\end{aligned}
$$

For $n \geq 1$,

$$
\mathrm{e}^{-\beta h_{1,n}} = \begin{pmatrix} |g\rangle\,|n\rangle & |e\rangle\,|n-1\rangle \end{pmatrix} \begin{pmatrix} \cosh(\beta a_n) + \frac{m}{a_{1,n}}\sinh(\beta a_{1,n}) & \frac{\sqrt{n}}{\lambda a_{1,n}}\sinh(\beta a_{1,n}) \\ \frac{\sqrt{n}}{\lambda a_{1,n}}\sinh(\beta a_{1,n}) & \cosh(\beta a_{1,n}) - \frac{m}{a_{1,n}}\sinh(\beta a_{1,n}) \end{pmatrix} \begin{pmatrix} \langle g|\,\langle n| \\ \langle e|\,\langle n-1| \end{pmatrix}
$$

First, we consider the state with the positive energy.

$$
U\mathrm{e}^{-\beta H}U^\dagger = \begin{pmatrix} \mathrm{e}^{-\beta h_1} & 0 \\ 0 & \mathrm{e}^{-\beta h_2} \end{pmatrix}
$$

The projection to positive energy state is expressed as

$$
P_{e^-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Then, UP_{e^-} and $P_{e^-}U^\dagger = P_{e^-}U$ are

$$UP_{e^-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P_{e^-}U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$U\mathrm{e}^{-\beta H}U^\dagger =$

$$\begin{pmatrix} \cosh(\beta a_n) + \frac{m}{a_n}\sinh(\beta a_n) & \frac{\sqrt{n}}{\lambda a_n}\sinh(\beta a_n) & 0 & 0 \\ \frac{\sqrt{n}}{\lambda a_n}\sinh(\beta a_n) & \cosh(\beta a_n) - \frac{m}{a_n}\sinh(\beta a_n) & 0 & 0 \\ 0 & 0 & \cosh(\beta a_n) - \frac{m}{a_n}\sinh(\beta a_n) & \frac{\sqrt{n}}{\lambda a_n}\sinh(\beta a_n) \\ 0 & 0 & \frac{\sqrt{n}}{\lambda a_n}\sinh(\beta a_n) & \cosh(\beta a_n) + \frac{m}{a_n}\sinh(\beta a_n) \end{pmatrix}$$

$$\begin{aligned} P_{e^-}\mathrm{e}^{-\beta H}P_{e^-} &= P_{e^-}U^\dagger U\mathrm{e}^{-\beta H}U^\dagger UP_{e^-} \\ &= \begin{pmatrix} \cosh(\beta a_n) + \frac{m}{a_n}\sinh(\beta a_n) & 0 & 0 & 0 \\ 0 & \cosh(\beta a_n) + \frac{m}{a_n}\sinh(\beta a_n) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the effective positive energy state ρ_{e^-} is

$$\rho_{e^-} = Z_{e^-}^{-1}\{\cosh(\beta a_n) + \frac{m}{a_n}\sinh(\beta a_n)\}(|0\rangle\langle 0| \otimes |n\rangle\langle n| + |1\rangle\langle 1| \otimes |n-1\rangle\langle n-1|) \tag{G.9}$$

We take partial trace $P_{e^-}\mathrm{e}^{-\beta H}P_{e^-}$ with respect to $|0\rangle$ and $|1\rangle$

$$\begin{aligned} \mathrm{tr}_{\mathrm{spin}}\,[P_{e^-}\mathrm{e}^{-\beta H}P_{e^-}] &= \sum_{n=0}\cosh(\beta a_n)\,|n\rangle\langle n| + \sum_{n=1}\frac{m}{a_n}\sinh(\beta a_n)\,|n-1\rangle\langle n-1| \\ &= \sum_{n=0}\cosh(\beta a_n)\,|n\rangle\langle n| + \sum_{n=0}\frac{m}{a_{n+1}}\sinh(\beta a_{n+1})\,|n\rangle\langle n| \\ &= \sum_{n=0}\{\cosh(\beta a_n) + \frac{m}{a_{n+1}}\sinh(\beta a_{n+1})\}\,|n\rangle\langle n| \end{aligned} \tag{G.10}$$

Below is the calculation including the negative energy state.

Partial trace $\mathrm{e}^{-\beta h_1}$ with respect to $|e\rangle$ and $|g\rangle$.

$$\begin{aligned}\mathrm{tr}_{\mathrm{spin}}[\sum_{n=0}\mathrm{e}^{-\beta h_{1,n}}] = |0\rangle\langle 0|\{\cosh(\beta a_{1,0}) + \frac{m}{a_{1,0}}\sinh(\beta a_{1,0})\} \\ + \sum_{n=1}[\{\cosh(\beta a_{1,n}) + \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}|n\rangle\langle n| \\ + \{\cosh(\beta a_{1,n}) - \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}|n-1\rangle\langle n-1|]\end{aligned}$$

$h_{2,n}$ is

$$h_{2,n} = P_n h_2 P_n = \frac{\sqrt{n}}{\lambda}\tau_1 - m\tau_3 = \vec{a_{2,n}}\,\vec{\tau}, \tag{G.11}$$

where $\vec{a_2} = (\frac{\sqrt{n}}{\lambda}, 0, -m)$.

$|\vec{a_2}| = a_{2,n} = \sqrt{\frac{n}{\lambda^2} + m^2} = a_{1,n}$.

Partial trace $\mathrm{e}^{-\beta h_2}$ with respect to $|e\rangle$ and $|g\rangle$.

$$\begin{aligned}\mathrm{tr}_{\mathrm{spin}}[\sum_{n=0}\mathrm{e}^{-\beta h_{1,n}}] = |0\rangle\langle 0|\{\cosh(\beta a_{1,0}) - \frac{m}{a_{1,0}}\sinh(\beta a_{1,0})\} \\ + \sum_{n=1}[\{\cosh(\beta a_{1,n}) - \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}|n\rangle\langle n| \\ + \{\cosh(\beta a_{1,n}) + \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}|n-1\rangle\langle n-1|]\end{aligned} \tag{G.12}$$

Therefore $\mathrm{tr}_{\mathrm{spin}}[\mathrm{e}^{-\beta h_1} + \mathrm{e}^{-\beta h_2}]$ is

$$
\begin{aligned}
\mathrm{tr}_{\mathrm{spin}}[\mathrm{e}^{-\beta h_1} + \mathrm{e}^{-\beta h_2}] &= 2\,|0\rangle\langle 0|\cosh(\beta a_{1,0}) \\
&+ \sum_{n=1}[\{\cosh(\beta a_{1,n}) + \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}\,|n\rangle\langle n| \\
&+ \{\cosh(\beta a_{1,n}) - \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}\,|n-1\rangle\langle n-1|] \\
&+ \sum_{n=1}[\{\cosh(\beta a_{1,n}) - \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}\,|n\rangle\langle n| \\
&+ \{\cosh(\beta a_{1,n}) + \frac{m}{a_{1,n}}\sinh(\beta a_{1,n})\}\,|n-1\rangle\langle n-1|] \\
&= 2\,|0\rangle\langle 0|\cosh(\beta a_{1,0}) + 2\sum_{n=1}\{\cosh(\beta a_{1,n})\,|n\rangle\langle n| + \cosh(\beta a_{1,n})\,|n-1\rangle\langle n-1|\} \\
&= 2\cosh(\beta a_{1,0})\,|0\rangle\langle 0| \\
&+ 2\cosh(\beta a_{1,1})\,|0\rangle\langle 0| \\
&+ 2\cosh(\beta a_{1,1})\,|1\rangle\langle 1| + 2\cosh(\beta a_{1,2})\,|1\rangle\langle 1| \\
&+ 2\cosh(\beta a_{1,2})\,|2\rangle\langle 2| + 2\cosh(\beta a_{1,3})\,|2\rangle\langle 2| \\
&+ 2\cosh(\beta a_{1,3})\,|3\rangle\langle 3| + \cdots \\
&= 2\sum_{n=0}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\}\,|n\rangle\langle n|
\end{aligned}
$$

Define the reference state ρ_0 as

$$
\rho_0 = Z_\beta^{-1}\mathrm{tr}_{\mathrm{spin}}[\mathrm{e}^{-\beta h_1} + \mathrm{e}^{-\beta h_2}]
$$

so that the thermal state can be the reference state. Therefore,

$$
Z_\beta^{-1} = \mathrm{tr}[\mathrm{e}^{-\beta h_1} + \mathrm{e}^{-\beta h_2}] = 2\sum_{n=0}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\}
$$

Then, the reference state ρ_0 is

$$
\rho_0 = \frac{\sum_{n=0}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\}\,|n\rangle\langle n|}{\sum_{n=0}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\}} \tag{G.13}
$$

where $a_{1,n} = \sqrt{\frac{n}{\lambda^2} + m^2}$.

G.3 RLD Fisher information

G.3.1 Unitary tranformation

$$
\rho_\theta = \mathrm{e}^{-\mathrm{i}\Pi^2\theta^2}\mathrm{e}^{-\mathrm{i}\Pi^1\theta^1}\rho_0\mathrm{e}^{\mathrm{i}\Pi^1\theta^1}\mathrm{e}^{\mathrm{i}\Pi^2\theta^2} \tag{G.14}
$$

With using

$$\Pi^1 = \frac{1}{\lambda}(\alpha + \alpha^\dagger),$$
$$\Pi^2 = \frac{\mathrm{i}}{\lambda}(\alpha^\dagger - \alpha),$$

we obtain

$$\rho_\theta = \mathrm{e}^{\frac{1}{\lambda}(\alpha-\alpha^\dagger)\theta^2}\mathrm{e}^{-\frac{\mathrm{i}}{\lambda}(\alpha+\alpha^\dagger)\theta^1}\rho_0\mathrm{e}^{\frac{\mathrm{i}}{\lambda}(\alpha+\alpha^\dagger)\theta^1}\mathrm{e}^{-\frac{1}{\lambda}(\alpha-\alpha^\dagger)\theta^2}. \tag{G.15}$$

G.3.2 Commutation relation between the annihilation creation operators and the reference state

To obtain RLD, L_0^R, we need the commutation relation $[\alpha, \rho_0]$ and $[\alpha^\dagger, \rho_0]$. To derive the commutation relation, we first introduce the function of the operator $\alpha^\dagger\alpha = N$, $f(N)$ which has the same form of the reference state ρ_0 as

$$f(N) = \sum_n^\infty f(n)\left|n\right\rangle\left\langle n\right| \tag{G.16}$$

where $N = \alpha^\dagger\alpha$. The validity of the equation above is confirmed as

$$\begin{aligned}
\because f(N) &= \mathrm{I}f(N)\mathrm{I} \\
&= \sum_n^\infty\sum_m^\infty \left|n\right\rangle\left\langle n\right| f(N)\left|m\right\rangle\left\langle m\right| \\
&= \sum_n^\infty\sum_m^\infty \left|n\right\rangle\left\langle n\right| f(m)\left|m\right\rangle\left\langle m\right| \\
&= \sum_n^\infty\sum_m^\infty f(m)\left|n\right\rangle\left\langle n|m\right\rangle\left\langle m\right| \\
&= \sum_n^\infty\sum_m^\infty f(m)\left|n\right\rangle\delta_{n,m}\left\langle m\right| \\
&= \sum_n^\infty f(n)\left|n\right\rangle\left\langle n\right|
\end{aligned} \tag{G.17}$$

By the definition, $f(N)$ is

$$f(N) = f(\alpha^\dagger\alpha) = \sum_n^\infty C_n(\alpha^\dagger\alpha)^n \tag{G.18}$$

Prove $\alpha(\alpha^\dagger\alpha)^n = (\alpha^\dagger\alpha)^n\alpha$ based on the mathematical induction in the following.

For $n = 1$,

$$\alpha(\alpha^\dagger\alpha) = (\alpha\alpha^\dagger)\alpha = (\alpha^\dagger\alpha + 1)\alpha \tag{G.19}$$

Next, by assuming that $\alpha(\alpha^\dagger\alpha)^{n-1} = (\alpha^\dagger\alpha + 1)^{n-1}\alpha$ holds, prove $\alpha(\alpha^\dagger\alpha)^n = (\alpha^\dagger\alpha + 1)^n\alpha$.

$$\alpha(\alpha^\dagger\alpha)^n = \alpha(\alpha^\dagger\alpha)(\alpha^\dagger\alpha)^{n-1} = (\alpha^\dagger\alpha + 1)\alpha(\alpha^\dagger\alpha)^{n-1} \tag{G.20}$$

By the assumption, $\alpha(\alpha^\dagger\alpha)^{n-1} = (\alpha^\dagger\alpha + 1)^{n-1}\alpha$

$$\therefore \alpha(\alpha^\dagger\alpha)^n = (\alpha^\dagger\alpha + 1)(\alpha^\dagger\alpha + 1)^{n-1}\alpha = (\alpha^\dagger\alpha + 1)^n\alpha \tag{G.21}$$

Then,

$$\alpha f(N) = \alpha\sum_n^\infty C_n(\alpha^\dagger\alpha)^n = \sum_n^\infty C_n(\alpha^\dagger\alpha + 1)^n\alpha = f(N+1)\alpha \tag{G.22}$$

The Hermite conjugate of the equation above is

$$f(N)\,\alpha^\dagger = \alpha^\dagger f(N+1) \tag{G.23}$$

$$\therefore \alpha^\dagger f(N) = f(N-1)\,\alpha^\dagger \tag{G.24}$$

If $\rho_0 = f(N) = \sum_n^\infty f(n)\,|n\rangle\langle n|$ holds,

$$\rho_0 = f(N) = \sum_n^\infty f(n)\,|n\rangle\langle n| > 0 \tag{G.25}$$

$$\rho_0^{-1} = \sum_n^\infty \frac{1}{f(n)}\,|n\rangle\langle n| \tag{G.26}$$

RLD, L_0^R is defined as

$$\frac{d\rho_\theta}{d\theta} = \rho_\theta L^R \tag{G.27}$$

$$\therefore L_{0,i}^R = \rho_0^{-1}\frac{d}{d\theta^i}\rho_\theta\,|_{\theta^i=0}$$

$$\begin{aligned}
L_{0,1}^R &= \rho_0^{-1}\frac{-\mathrm{i}}{\lambda}\{(\alpha+\alpha^\dagger)\rho_0 - \rho_0(\alpha+\alpha^\dagger)\} \\
&= \frac{\mathrm{i}}{\lambda}\{(\alpha+\alpha^\dagger) - \rho_0^{-1}(\alpha+\alpha^\dagger)\rho_0\} \\
&= \frac{\mathrm{i}}{\lambda}\{(\alpha+\alpha^\dagger) - \rho_0^{-1}(f(N+1)\alpha + f(N-1)\alpha^\dagger)\} \\
&= \frac{\mathrm{i}}{\lambda}\{(\alpha+\alpha^\dagger) - \rho_0^{-1}(f(N+1)\alpha + f(N-1)\alpha^\dagger)\}
\end{aligned}$$

$\rho_0^{-1}(f(N+1)$ is calculated as follows.

$$\begin{aligned}
\rho_0^{-1} f(N+1) &= \sum_{n=0}^{\infty} \frac{1}{f(n)} |n\rangle \langle n| \sum_{m=0}^{\infty} f(m+1) |m\rangle \langle m| \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{f(n)} f(m+1) |n\rangle \langle n|m\rangle \langle m| \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{f(n)} f(m+1) |n\rangle \delta_{n,m} \langle m| \\
&= \sum_{m=0}^{\infty} \frac{f(n+1)}{f(n)} |n\rangle \langle n|
\end{aligned}$$

$\rho_0^{-1}(f(N-1)$ is calculated as follows.

$$\begin{aligned}
\rho_0^{-1} f(N-1) &= \sum_{n=0}^{\infty} \frac{1}{f(n)} |n\rangle \langle n| \sum_{m=1}^{\infty} f(m-1) |m\rangle \langle m| \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{f(n)} f(m-1) |n\rangle \langle n|m\rangle \langle m| \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{f(n)} f(m-1) |n\rangle \delta_{n,m} \langle m| \\
&= \sum_{n=1}^{\infty} \frac{f(n-1)}{f(n)} |n\rangle \langle n|
\end{aligned}$$

Then, $L^R_{0,1}$ is calculated as

$$\begin{aligned}
L^R_{0,1} &= \frac{\mathrm{i}}{\lambda}\{(\alpha + \alpha^\dagger) - \sum_{m=0}^{\infty} \frac{f(n+1)}{f(n)} |n\rangle \langle n| \alpha - \sum_{n=1}^{\infty} \frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha + \alpha^\dagger - \sum_{n=1}^{\infty} \frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha + \sum_{n=0}^{\infty} |n\rangle \langle n| \alpha^\dagger - \sum_{n=1}^{\infty} \frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha + (|0\rangle \langle 0| + \sum_{n=1}^{\infty} |n\rangle \langle n|)\alpha^\dagger - \sum_{n=1}^{\infty} \frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha + |0\rangle \langle 0| \alpha^\dagger - \sum_{n=1}^{\infty}(\frac{f(n-1)}{f(n)} - 1) |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha - \sum_{n=1}^{\infty}(\frac{f(n-1)}{f(n)} - 1) |n\rangle \langle n| \alpha^\dagger\} \\
&= \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty} \frac{f(n) - f(n+1)}{f(n)} |n\rangle \langle n| \alpha + \sum_{n=1}^{\infty} \frac{f(n) - f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\}
\end{aligned}$$

By using the following relation,

$$|n\rangle \langle n| \alpha = \sqrt{n+1}\, |n\rangle \langle n+1|, \quad |n\rangle \langle n| \alpha^\dagger = \sqrt{n}\, |n\rangle \langle n-1|,$$

we get

$$L^R_{0,1} = \frac{\mathrm{i}}{\lambda}\{\sum_{n=0}^{\infty} \sqrt{n+1}\frac{f(n)+f(n+1)}{f(n)} |n\rangle \langle n+1| - \sum_{n=1}^{\infty} \sqrt{n}\frac{f(n)-f(n-1)}{f(n)} |n\rangle \langle n-1|\}.$$

From (G.15) and $\rho_0 = \sum_n f(n) |n\rangle \langle n|$

$$f(n) = Z_\beta^{-1}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\} \tag{G.28}$$

where $Z_\beta = \sum_{n=0}\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\} = \cosh(\beta a_{1,0}) + 2\sum_{n=1}\{\cosh(\beta a_{1,n})\}$.

$$\begin{aligned} f(n+1) - f(n) &= Z_\beta^{-1}[\{\cosh(\beta a_{1,n+1}) + \cosh(\beta a_{1,n+2})\} - \{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\}] \\ &= Z_\beta^{-1}\{\cosh(\beta a_{1,n+2}) - \cosh(\beta a_{1,n})\} \end{aligned}$$

$$\begin{aligned} f(n) - f(n-1) &= Z_\beta^{-1}[\{\cosh(\beta a_{1,n}) + \cosh(\beta a_{1,n+1})\} - \{\cosh(\beta a_{1,n-1}) + \cosh(\beta a_{1,n})\}] \\ &= Z_\beta^{-1}\{\cosh(\beta a_{1,n+1}) - \cosh(\beta a_{1,n-1})\} \end{aligned} \tag{G.29}$$

$$\begin{aligned} g_{11} &= \mathrm{tr}[\rho_0 L^R_{0,1} L^{R,\dagger}_{0,1}] \\ &= \frac{1}{\lambda^2}\{\sum_{n=0}^{\infty}[(n+1)f(n)\{\frac{f(n)-f(n+1)}{f(n)}\}^2 + nf(n)\{\frac{f(n)-f(n-1)}{f(n)}\}^2] \end{aligned}$$

$$\begin{aligned} L^R_{0,2} &= \rho_0^{-1}\frac{1}{\lambda}\{(\alpha-\alpha^\dagger)\rho_0 - \rho_0(\alpha-\alpha^\dagger)\} \\ &= -\frac{1}{\lambda}\{(\alpha-\alpha^\dagger) - \rho_0^{-1}(\alpha-\alpha^\dagger)\rho_0\} \\ &= -\frac{1}{\lambda}\{(\alpha-\alpha^\dagger) - \rho_0^{-1}(f(N+1)\alpha - f(N-1)\alpha^\dagger)\} \\ &= -\frac{1}{\lambda}\{(\alpha-\alpha^\dagger) - \rho_0^{-1}(f(N+1)\alpha + f(N-1)\alpha^\dagger)\} \end{aligned}$$

Then, $L^R_{0,2}$ is calculated as

$$\begin{aligned} L^R_{0,2} &= -\frac{1}{\lambda}\{(\alpha-\alpha^\dagger) - \sum_{m=0}^{\infty}\frac{f(n+1)}{f(n)} |n\rangle \langle n| \alpha + \sum_{n=1}^{\infty}\frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\ &= -\frac{1}{\lambda}\{\sum_{n=0}^{\infty}(1 - \frac{f(n+1)}{f(n)}) |n\rangle \langle n| \alpha - \alpha^\dagger + \sum_{n=1}^{\infty}\frac{f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\ &= -\frac{1}{\lambda}\{\sum_{n=0}^{\infty}\frac{f(n)-f(n+1)}{f(n)} |n\rangle \langle n| \alpha - \sum_{n=1}^{\infty}\frac{f(n)-f(n-1)}{f(n)} |n\rangle \langle n| \alpha^\dagger\} \\ &= -\frac{1}{\lambda}\{\sum_{n=0}^{\infty} \sqrt{n+1}\frac{f(n)-f(n+1)}{f(n)} |n\rangle \langle n+1| - \sum_{n=1}^{\infty} \sqrt{n}\frac{f(n)-f(n-1)}{f(n)} |n\rangle \langle n-1|\} \end{aligned}$$

RLD fisher information is calculated as follows.

$$\begin{aligned} g_{11} &= \mathrm{tr}[\rho_0 L^R_{0,1} L^{R,\dagger}_{0,1}] \\ &= \frac{1}{\lambda^2}\{\sum_{n=0}^{\infty}[(n+1)f(n)\{\frac{f(n)-f(n+1)}{f(n)}\}^2 + \sum_{n=1}^{\infty} nf(n)\{\frac{f(n)-f(n-1)}{f(n)}\}^2] \end{aligned}$$

Let $n = n' + 1$ in the second term above. Then,

$$\begin{aligned} \text{second term} &= \frac{1}{\lambda^2}\sum_{n'=0}^{\infty}(n'+1)\frac{\{f(n'+1)-f(n')\}^2}{f(n'+1)} \\ &= \frac{1}{\lambda^2}\sum_{n=0}^{\infty}(n+1)\frac{\{f(n+1)-f(n)\}^2}{f(n+1)} \end{aligned} \tag{G.30}$$

$$\therefore g_{11} = \frac{1}{\lambda^2}\sum_{n=0}^{\infty}(n+1)\{f(n+1)-f(n)\}^2\{\frac{1}{f(n)}+\frac{1}{f(n+1)}\}$$

$$\begin{aligned} g_{22} &= \mathrm{tr}[\rho_0 L^R_{0,2} L^{R\dagger}_{0,2}] \\ &= \frac{1}{\lambda^2}\sum_{n=0}^{\infty}(n+1)\{f(n+1)-f(n)\}^2\{\frac{1}{f(n)}+\frac{1}{f(n+1)}\} = g_{11} \end{aligned}$$

$$\begin{aligned} g_{12} &= \mathrm{tr}[\rho_0 L^R_{0,2} L^{R\dagger}_{0,1}] \\ &= -\frac{\mathrm{i}}{\lambda^2}\sum_{n=0}^{\infty}(n+1)\{f(n+1)-f(n)\}^2\{\frac{1}{f(n)}-\frac{1}{f(n+1)}\} \end{aligned}$$

G^R has a form

$$G^R = \begin{pmatrix} A+B & -\mathrm{i}(A-B) \\ \mathrm{i}(A-B) & A+B \end{pmatrix}$$

Therefore, the inverse of G^R is

$$(G^R)^{-1} = \frac{1}{4AB}\begin{pmatrix} A+B & \mathrm{i}(A-B) \\ -\mathrm{i}(A-B) & A+B \end{pmatrix}$$

G.3.3 Positive energy state

$$f(n) = Z_{e^-}^{-1}\{\cosh(\beta a_n) + \frac{m}{a_{n+1}}\sinh(\beta a_{n+1})\} \tag{G.31}$$

The parameters that appear in the equations are

$\lambda = \sqrt{\frac{2}{eB}}$

$a_n = \sqrt{m^2 + \frac{n}{\lambda^2}} = \sqrt{m^2 + \frac{n}{\lambda^2}}$
Put $c, \hbar$ back into a_n.

$$\beta a_n = \beta mc^2 \sqrt{\frac{n\hbar c^2 eB}{2m^2c^4} + 1} = \beta mc^2 \sqrt{\frac{n\hbar\omega}{2mc^2} + 1} \tag{G.32}$$

We can evaluate the parameter βa_n as follows. We assume the magnetic field, $B = 1T$.
$\omega = \dfrac{eB}{m} = 1.76 \times 10^{-19}$ Hz
$\hbar\omega = 1.84 \times 10^{-23}$ J $= 1.15 \times 10^{-4}$ eV
$mc^2 = 0.5$ MeV
$\dfrac{\hbar\omega}{mc^2} = \frac{1.1\times10^{-4}}{0.5\times10^{3}} = 2.2 \times 10^{-7}$

At the room temperature, $kT = \frac{1}{40}$ eV
$\beta mc^2 = 40 \times 500 = 2.0 \times 10^4$
$\beta a_n = \beta mc^2 \sqrt{1 + \frac{n\hbar\omega}{2mc^2}} = 2 \times 10^4 \sqrt{1 + 2.2 \times 10^{-7} n} \fallingdotseq 2 \times 10^4$

Therefore, $f(n)$ in (G.31) does not converge.